PURE MATHEMATICS

Spring Semester 2000-2001 2 hours 30 minutes

Algebraic Topology

Answer ***four*** *questions. You are advised* ***not*** *to answer more than four questions: if you do, only your best four will be counted.*

1 (i) What does it mean to say that a metric space X is *compact*? ***(3 marks)***

(ii) Let $f\colon X \to Y$ be a continuous surjective map of metric spaces, where X is compact. Prove that Y is compact. ***(6 marks)***

(iii) Let Z be a closed subset of a compact space X. Prove that Z is compact. ***(6 marks)***

(iv) Put $U = \{z \in \mathbb{C} \mid 0 \leqslant \mathrm{Re}(z) \leqslant 1\}$, and define $g\colon \mathbb{C} \to \mathbb{C}$ by $g(z) = e^z$.

(a) Is U compact? ***(2 marks)***

(b) Is $g(U)$ compact? ***(4 marks)***

(c) Is $g(g(U))$ compact? ***(4 marks)***

Justify your answers.

2 (i) Let X be a metric space. Define the equivalence relation $\sim$ on X such that $\pi_0 X = X/\sim$, and prove that it is indeed an equivalence relation. ***(8 marks)***

(ii) Let $f\colon X \to Y$ be a continuous map. Define the function $f_*\colon \pi_0 X \to \pi_0 Y$, and check that it is well-defined. ***(5 marks)***

(iii) Suppose that Y is path-connected and X is not. Show that there do not exist maps $f\colon X \to Y$ and $g\colon Y \to X$ such that gf is homotopic to the identity map 1_X. ***(6 marks)***

(iv) Put $X = \{A \in M_2\mathbb{R} \mid A^2 = A\}$. What can you say about $\det(A)$ when $A \in X$? Show that X is not path-connected. ***(6 marks)***

3 Are the following statements true or false? Give proof or disproof as appropriate. You may quote general theorems without proof, provided that you state them clearly. You may give pictures instead of formulae, provided that they are clearly explained.

(i) There is a continuous surjective map from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R} \setminus \{0\}$ ***(5 marks)***

(ii) $S^2 \setminus S^1$ is homeomorphic to $\mathbb{R}^2$ ***(5 marks)***

(iii) $SO(2)$ is homotopy equivalent to the Möbius strip ***(5 marks)***

(iv) $SO(3)$ is homotopy equivalent to the torus ***(5 marks)***

(v) The space $X = S^1 \cup \{(x,0) \mid x \in \mathbb{R}\}$ is homeomorphic to $Y = S^1 \cup \{(x,1) \mid x \in \mathbb{R}\}$. ***(5 marks)***

4 Give examples of the following things.

(i) A space X and a point $x \in X$ such that X is not contractible but $X \setminus \{x\}$ is contractible. ***(3 marks)***

(ii) A subspace $X \subseteq \mathbb{R}^2$ that is homotopy equivalent to $S^4 \setminus S^2$. (You need not give a proof.) ***(4 marks)***

(iii) Spaces X and Y, a discontinuous map $f\colon X \to Y$, and an open subset $V \subseteq Y$ such that $f^{-1}V$ is not open in X. (You should justify your answer carefully.) ***(6 marks)***

(iv) A space X and a point $x \in X$ such that $\pi_1 X$ is abelian and $\pi_1(X \setminus \{x\})$ is nonabelian. (You should state what $\pi_1 X$ and $\pi_1(X \setminus \{x\})$ are, but no further justification is required.) ***(6 marks)***

(v) A space X such that $a(X) = 2$ and $b(X) = 2$, where as usual

$$\begin{aligned} a(X) &= \max\{|S| \mid S \text{ is a finite subset of } X \text{ and } X \setminus S \text{ is path-connected }\} \\ &= \text{the largest number of points that can be} \\ &\quad\ \text{removed from } X \text{ without disconnecting it} \\ b(X) &= \min\{|S| \mid S \text{ is a finite subset of } X \text{ and } X \setminus S \text{ is not path-connected }\} \\ &= \text{the smallest number of points that have to be} \\ &\quad\ \text{removed from } X \text{ to disconnect it} \end{aligned}$$

(You should justify your answer, but complete rigour is not required.) ***(6 marks)***

5 (i) Let X be a subspace of $\mathbb{R}^n$, and let a be a point in X.

(a) Explain what it means for X to be *star-shaped* around a.

(4 marks)

(b) Prove that if X is star-shaped around a, then X is contractible.

(4 marks)

(ii) (a) Suppose that $\alpha, \beta > 0$ and that $0 \leqslant t \leqslant 1$. Show that $\alpha t + \beta(1-t)$ is strictly greater than zero. ***(3 marks)***

(b) Suppose that $\gamma, \delta, \epsilon > 0$ and that $0 \leqslant t \leqslant 1$. Show that $\gamma t^2 + \delta t(1-t) + \epsilon(1-t)^2$ is strictly greater than zero. ***(3 marks)***

(c) Consider a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2\mathbb{R}$. Put $\lambda = \text{trace}(A)$ and $\mu = \det(A)$. Express $\text{trace}((1-t)I + tA)$ and $\det((1-t)I + tA)$ in terms of λ, μ and t. ***(6 marks)***

(d) Put $X = \{A \in M_2\mathbb{R} \mid \det(A) > 0 \text{ and } \text{trace}(A) > 0\}$. Prove that X is contractible. ***(5 marks)***

End of Question Paper