GUIDANCE ON THE EXAM

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1. Examples

This list covers most of the examples that you should know about. The exam may ask you to prove that two spaces are (or are not) homeomorphic (or homotopy equivalent) to each other. The spaces involved will either be taken from the list below, or they will be easily visualised subsets of $\mathbb{R}^2$ or $\mathbb{R}^3$.

1. The Euclidean spaces $\mathbb{R}^n$, particularly for $n = 1, 2, 3$. Some spaces derived from these like $\mathbb{R}^n \setminus \{0\}$, $\mathbb{R} \setminus \mathbb{Z}$, $\mathbb{R}^2 \setminus \{P, Q\}$, $\mathbb{R}^2 \setminus S^1$ and $\mathbb{R}^3 \setminus S^1$.
2. The spheres S^n, particularly for $n = 0, 1, 2, 3$. Some spaces derived from these such as $S^n \setminus \{N\}$ and $S^n \setminus S^m$ (where $m < n$).
3. The balls B^n.
4. The projective spaces $\mathbb{R}P^n$, particularly for $n = 1, 2, 3$.
5. The intervals $[a, b]$, $[a, b)$, $(a, b]$ and (a, b) where $a, b \in \mathbb{R}$ and $a < b$. The intervals $(-\infty, c]$, $(-\infty, c)$, $[c, \infty)$, (c, ∞) and $(-\infty, \infty)$, where $c \in \mathbb{R}$.
6. The matrix groups $SO(2)$, $O(2)$, $SL_2\mathbb{R}$, $SO(3)$, $SU(2)$.
7. The n-torus $(S^1)^n$, particularly the 2-torus $T = S^1 \times S^1$. The solid torus $S^1 \times B^2$ and the punctured torus $T \setminus \{(1, 1)\}$.
8. Letters of the alphabet, considered as subsets of $\mathbb{R}^2$. Other one-dimensional subsets of $\mathbb{R}^2$, such as

$$X = \{(x, y) \in \mathbb{R}^2 \mid xy = 0\} = (x\text{-axis}) \cup (y\text{-axis}).$$

9. The figure eight space E.
10. The Möbius strip M.
11. The standard orientable surface S_g of genus g (a doughnut with g holes).

Some homeomorphisms among these spaces are as follows.

(a) The spaces $\mathbb{R} = (-\infty, \infty)$, (a, b), (a, ∞) and $(-\infty, a)$ are all homeomorphic to each other. The main ingredient in the proof is the map $f\colon (-1, 1) \to \mathbb{R}$ given by $f(x) = x/(1 - x^2)$. If $a < b$ and $a' < b'$ then (a, b) is homeomorphic to (a', b'), and similarly for infinite intervals.
(b) The spaces $[a, b)$, $(a, b]$, $[a, \infty)$ and $(-\infty, a]$ are all homeomorphic to each other.
(c) The space $\mathbb{C}$ is homeomorphic to $\mathbb{R}^2$, and thus $\{z \in \mathbb{C} \mid |z| = 1\}$ is homeomorphic to S^1. These homeomorphisms are so basic that we usually do not bother to mention them, and just regard $\mathbb{R}^2$ as being the same as $\mathbb{C}$.
(d) For any point $a \in S^n$, the space $S^n \setminus \{a\}$ is homeomorphic to $\mathbb{R}^n$ by stereographic projection. In particular $S^1 \setminus \{a\}$ is homeomorphic to $\mathbb{R}$ and thus to all the other spaces mentioned in (a). Similarly, $S^2 \setminus \{a\}$ is homeomorphic to $\mathbb{R}^2$ and thus to $\mathbb{C}$.
(e) For any point $a \in \mathbb{R}^n$, the space $\mathbb{R}^n \setminus \{a\}$ is homeomorphic to $S^{n-1} \times (0, \infty)$.
(f) $SO(2)$ is homeomorphic to S^1, and $O(2)$ is homeomorphic to $S^1 \times \{1, -1\}$. $SL_2\mathbb{R}$ is homeomorphic to $S^1 \times (0, \infty) \times \mathbb{R}$. $SO(3)$ is homeomorphic to $\mathbb{R}P^3$, and homotopy equivalent to $GL_3(\mathbb{R})$. $SU(2)$ is homeomorphic to S^3.

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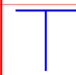
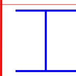
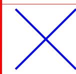
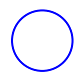
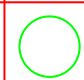
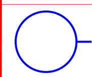
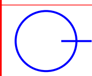
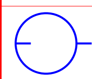
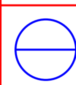
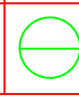
(g) The letters of the alphabet fall into nine different homeomorphism classes, as indicated by the lines in the following table:

		a	b	
CLMNSUVWZ		2	1	
YTJGEF		3	1	
HI		4	1	
KX		4	1	
OD		1	2	
P		2	1	
Q		3	1	
AR		3	1	
B		2	2	

(h) $\mathbb{R}P^1$ is homeomorphic to S^1.

Of course, if two spaces are homeomorphic, they are also homotopy equivalent. In addition to this, we have some further homotopy equivalences:

(A) All the spaces in (a), (b) and (d) above are contractible, or in other words, homotopy equivalent to a single point.
(B) $\mathbb{R}^n \setminus \{a\}$ is homotopy equivalent to S^{n-1}.
(C) $S^n \setminus S^m$ is homotopy equivalent to S^{n-m-1}.
(D) $\mathbb{R}^2 \setminus \{(0,0)\}$, $\mathbb{C} \setminus \{0\}$, the Möbius strip, the solid torus $S^1 \times B^2$, and the matrix groups $SO(2)$ and $SL_2\mathbb{R}$ are all homotopy equivalent to S^1.
(E) If P and Q are any two distinct points in $\mathbb{R}^2$, then $\mathbb{R}^2 \setminus \{P,Q\}$ is homotopy equivalent to the figure eight space E. This is also homotopy equivalent to the punctured torus $T \setminus \{(1,1)\}$.
(F) The letters O, D, P, Q, A and R are all homotopy equivalent to S^1, and the letter B is homotopy equivalent to the figure eight. All other letters are contractible.

The basic examples of fundamental groups are as follows:

- $\pi_1 S^1 = \pi_1 \mathbb{R}P^1 = \mathbb{Z}$
- $\pi_1 S^n = 0$ if $n > 1$
- $\pi_1 \mathbb{R}P^n = \mathbb{Z}/2$ if $n > 1$
- $\pi_1 E$ is a free group on two generators (which is infinite and nonabelian)
- $\pi_1(X \times Y) = \pi_1(X) \times \pi_1(Y)$; in particular, the torus $T = S^1 \times S^1$ has $\pi_1 T = \mathbb{Z} \times \mathbb{Z}$.
- If X is homotopy equivalent to X' then $\pi_1 X$ is isomorphic to $\pi_1 X'$.

2. Bookwork

The exam contains various questions that ask you to reproduce definitions, theorems or proofs. The topics are taken from the following list:

1. Metric space axioms
2. convergence
3. product spaces; convergence

4. continuity in terms of sequences
5. continuous maps into products
6. composition of continuous maps
7. $\epsilon - \delta$ continuity
8. open and closed sets; complementarity
9. unions and intersections, products
10. continuity and preimages
11. open patching and closed patching
12. compactness
13. I is compact; compactness of subsets of $\mathbb{R}^n$
14. products, closed subspaces and images
15. compact subsets are closed
16. continuous bijections with compact source
17. uniform continuity
18. distance for matrices
19. paths: join, reversal, constants
20. equivalence relations and π_0
21. linear paths and great circles
22. star shaped and convex sets
23. the intermediate value theorem and π_0
24. functorality of π_0
25. cutting numbers $a(X)$ and $b(X)$; invariance
26. homotopy; this gives an equivalence relation
27. linear homotopies
28. products and composites of homotopy classes
29. homotopy equivalence; this gives an equivalence relation
30. contractible spaces
31. homotopy invariance of π_0
32. path homotopy rel endpoints; equivalence relation
33. join and reversal, associativity and cancellation
34. the fundamental group
35. change of basepoint
36. functorality
37. products

More explicitly, the following items from the printed notes may be covered:

- Definition 2.1
- Definition 2.2
- Proposition 2.12
- Proposition 2.19
- Definition 3.1
- Proposition 3.6
- Proposition 3.9
- Definition 4.1
- Proposition 4.10
- Proposition 4.12
- Proposition 4.13
- Theorem 4.15
- Theorem 4.17
- Definition 7.2
- Proposition 7.5
- Proposition 7.13
- Proposition 7.14
- Definition 8.1
- Example 8.2
- Definition 8.4
- Definition 8.6
- Proposition 8.10
- Definition 8.13

- Proposition 8.14
- Proposition 8.17
- Definition 9.8
- Proposition 9.9
- Definition 10.1
- Proposition 10.4
- Definition 10.7
- Example 10.10
- Definition 10.11
- Example 10.12
- Proposition 10.18
- Proposition 12.6