PMA333 EXAMINATION SPRING 2001 — SOLUTIONS

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(1) (i) A metric space X is compact if for every sequence (x_n) in X there is a subsequence (x_{n_k}) and a point $x \in X$ such that $x_{n_k} \to x$.

(ii) Let $f\colon X \to Y$ be a continuous surjective map, and suppose that X is compact. Consider a sequence (y_n) in Y. As f is surjective, we can choose $x_n \in X$ for each n such that $f(x_n) = y_n$. As X is compact, there is a subsequence (x_{n_k}) of (x_n) and a point $x \in X$ such that $x_{n_k} \to x$. Put $y = f(x) \in Y$, and note that $y_{n_k} = f(x_{n_k})$. As f is continuous, it follows that $y_{n_k} \to y$. Thus our original sequence has a convergent subsequence, proving that Y is compact.

(iii) Let X be compact, and let Z be a closed subspace of X. Consider a sequence (z_n) in Z. We can regard this as a sequence in the compact space X, so some subsequence (z_{n_k}) converges to some point $x \in X$. However, Z is closed and z_{n_k} lies in Z for all k and $z_{n_k} \to x$, so x must actually lie in Z. Thus our original sequence has a subsequence that converges to a point in Z, proving that Z is compact.

(iv) Put $U = \{z \in \mathbb{C} \mid 0 \leq \mathrm{Re}(z) \leq 1\}$, and define $g\colon \mathbb{C} \to \mathbb{C}$ by $g(z) = e^z$. Then U is clearly unbounded and thus not compact; the sequence $i, 2i, 3i, \dots$ has no convergent subsequence. On the other hand, we can use the fact that $g(x+iy) = e^x(\cos(y) + i\sin(y))$ to see that $g(U) = \{z \in \mathbb{C} \mid 1 \leq |z| \leq e\}$. This is bounded and closed and thus compact . We can regard g as a continuous surjective map from $g(U)$ to $g(g(U))$ and it follows from (ii) that $g(g(U))$ is compact .

(2) (i) Write $x \sim y$ iff there is a path in X from x to y , or in other words a continuous map $u\colon I \to X$ such that $u(0) = x$ and $u(1) = y$. I claim that this is an equivalence relation. Indeed, given $x \in X$ we can define $c_x\colon I \to X$ by $c_x(t) = x$ for all t. This gives a path from x to itself, showing that $\sim$ is reflexive . Next, suppose that $x \sim y$, so there exists a path u from x to y in X. We can then define $\overline{u}(t) = u(1-t)$ to get a path from y to x, showing that $y \sim x$, showing that $\sim$ is symmetric . Finally, suppose we have a path u from x to y, and a path v from y to z. We then define a map $w\colon I \to X$ by

$$w(t) = \begin{cases} u(2t) & \text{if } 0 \leq t \leq 1/2 \\ v(2t-1) & \text{if } 1/2 \leq t \leq 1. \end{cases}$$

This is well-defined and continuous because $u(1) = y = v(0)$. We have $w(0) = u(0) = x$ and $w(1) = v(1) = z$, so w gives a path from x to z; this proves that $\sim$ is transitive .

(ii) Let $f\colon X \to Y$ be a continuous map. We define $f_*\colon \pi_0 X \to \pi_0 Y$ by $f_*([x]) = [f(x)]$ (where $[x]$ is the equivalence class of x under the relation $\sim$). To see that this is well-defined, suppose that $[x_0] = [x_1]$ in $\pi_0 X$. This means that $x_0 \sim x_1$, so there is a path $u\colon I \to X$ from x_0 to x_1 . The function $f \circ u\colon I \to Y$ gives a path from $f(x_0)$ to $f(x_1)$ in Y , so $[f(x_0)] = [f(x_1)]$ as required .

(iii) Suppose that Y is path-connected, so $\pi_0 Y$ has only a single element, which we will call b. Then $f_*\colon \pi_0 X \to \pi_0 Y$ must be the constant map with value b, so $g_* f_*\colon \pi_0 X \to \pi_0 X$ must be the constant map with value $g_*(b)$. On the other hand, if $gf \simeq 1$ then $g_* f_*$ is the identity. Thus, the identity map of $\pi_0 X$ is constant, so $\pi_0 X$ can only have a single element. This means that X is path-connected, contrary to assumption.

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(iv) Put $X = \{A \in M_2\mathbb{R} \mid A^2 = A\}$. For $A \in X$ we have $\det(A)^2 = \det(A)$ so $\det(A) \in \{0, 1\}$. We can thus regard det as a continuous map $X \to \mathbb{R}$ such that $\det(A) \neq 1/2$ for all A. The zero matrix and the identity matrix lie in X, with $\det(0) = 0 < 1/2$ and $\det(I) = 1 > 1/2$. It follows that 0 cannot be connected to I by a path in X, so X is not path-connected.

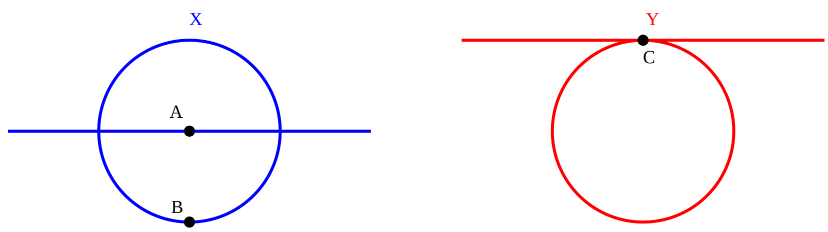
(3) (i) False. The space $\mathbb{R}^2$ is path-connected and $\mathbb{R} \setminus \{0\}$ is not, so there cannot be a continuous surjective map from $\mathbb{R}^2$ to $\mathbb{R} \setminus \{0\}$.

(ii) False. The space $S^2 \setminus S^1$ is homotopy equivalent to S^0, and thus is not path-connected; but $\mathbb{R}^2$ is evidently path-connected, by linear paths.

(iii) True. $SO(2)$ is homeomorphic to S^1 , and the Möbius strip M is homotopy equivalent to the circle running along the middle of the strip , so $SO(2)$ is homotopy equivalent to M.

(iv) False. We know that $SO(3)$ is homeomorphic to $\mathbb{R}P^3$, so $\pi_1 SO(3)$ has order two. On the other hand, $\pi_1 T \simeq \mathbb{Z} \times \mathbb{Z}$ is infinite, and thus not isomorphic to $\pi_1 SO(3)$, so T is not homotopy equivalent to $SO(3)$.

(v) False. The picture is as follows:



The points A and B can be removed from X without disconnecting it; but removing any two points from Y disconnects it. (Note in particular that removing C already disconnects Y).

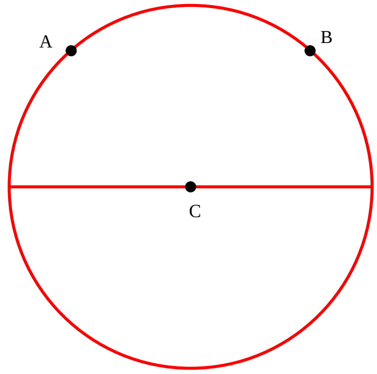
(4) (i) S^1 is not contractible (because $\pi_1 S^1 = \mathbb{Z}$ is nontrivial) but $S^1 \setminus \{1\}$ is homeomorphic to $\mathbb{R}$ and thus is contractible.

(ii) In general, $S^n \setminus S^m$ is homotopy equivalent to S^{n-m-1}. In particular, the space $S^4 \setminus S^2$ is homotopy equivalent to S^1, which is a subset of $\mathbb{R}^2$.

(iii) Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 0$ for $x \leq 0$ and $f(x) = 1$ for $x > 0$. This is discontinuous at $x = 0$, because $1/n \to 0$ but $f(1/n) = 1 \not\to 0 = f(0)$. If we put $V = (-1, 1) \subset \mathbb{R}$ then $f^{-1}V = (-\infty, 0]$. Thus V is open but $f^{-1}V$ is not .

(iv) Put $X = T = S^1 \times S^1$ and $x = (1, 1)$. Then $\pi_1 X = \mathbb{Z} \times \mathbb{Z}$, which is abelian. However, $X \setminus \{x\}$ is homotopy equivalent to a figure eight , so $\pi_1(X \setminus \{x\})$ is the free group on two generators , which is not abelian .

(v) We can take X to be the letter B, or the following space, which is homeomorphic to the letter B:



It is clear that $X \setminus \{A, C\}$ is connected (so $a(X) \geq 2$) and $X \setminus \{A, B\}$ is not (so $b(X) \leq 2$). By inspection, if we remove any one point, then X remains connected, so $b(X) = 2$. Also, if we remove any three points, then X becomes disconnected, so $a(X) = 2$.

(5) (i) (a) We say that X is star-shaped around a if for each $t \in I$ and $x \in X$, the point $(1-t)x + ta$ lies in X. Equivalently, X is star-shaped around a if every linear path starting in X and ending at a lies wholly in X.

(b) Suppose that X is star-shaped around a. We can then define a map $h\colon I \times X \to X$ by $h(t,x) = (1-t)x + ta$. We have $h(0,x) = x$ and $h(1,x) = a$ for all $x \in X$, so this gives a contraction of X.

(ii) (a) Suppose that $\alpha, \beta > 0$ and $0 \leq t \leq 1$. Then αt and $\beta(1-t)$ are both greater than or equal to 0. Moreover, αt is only zero when $t = 0$, and $\beta(1-t)$ is only zero when $t = 1$. Thus, for any t, at least one of the two terms is strictly positive, and thus $\alpha t + \beta(1-t) > 0$.

(b) Suppose that $\gamma, \delta, \epsilon > 0$ and that $0 \leq t \leq 1$. Then γt^2, $\delta t(1-t)$ and $\epsilon(1-t)^2$ are all greater than or equal to zero. The first one is strictly greater than zero unless $t = 0$, and the last one is strictly greater than zero unless $t = 1$. Thus, for all t, at least one term is strictly positive, so their sum is strictly positive.

(c) We have $\lambda = a + d$ and $\mu = ad - bc$ and

$$(1-t)I + tA = \begin{pmatrix} 1-t+ta & tb \\ tc & 1-t+td \end{pmatrix},$$

so

$$\begin{aligned} \operatorname{trace}((1-t)I + tA)) &= (ta + 1 - t) + (td + 1 - t) \\ &= (a+d)t + 2(1-t) \\ &= \lambda t + 2(1-t) \end{aligned}$$

and

$$\begin{aligned} \det((1-t)I + tA)) &= (ta + 1 - t)(td + 1 - t) - t^2 bc \\ &= (ad - bc)t^2 + (a+d)t(1-t) + (1-t)^2 \\ &= \mu t^2 + \lambda t(1-t) + (1-t)^2. \end{aligned}$$

(d) Suppose that $A \in X$, so $\lambda, \mu > 0$, and suppose that $t \in I$. As $\lambda > 0$ and $2 > 0$, part (a) tells us that $\lambda t + 2(1-t) > 0$, so $\operatorname{trace}((1-t)I + tA) > 0$. As $\mu > 0$, $\lambda > 0$ and $1 > 0$, part (b) tells us that $\mu t^2 + \lambda t(1-t) + (1-t)^2 > 0$, so $\det((1-t)I + tA) > 0$. This shows that $(1-t)I + tA \in X$, so X is star-shaped around I, and thus contractible.