UNIVERSITY OF SHEFFIELD — PMA333

PURE MATHEMATICS

Spring Semester 2000-2001 — 2 hours 30 minutes

Algebraic Topology Sample Exam

Answer ***four*** *questions. You are advised* ***not*** *to answer more than four questions: if you do, only your best four will be counted.*

1 (i) What does it mean to say that a metric space X is *path-connected*?

(ii) Prove that the space S^n is path-connected for all $n > 0$.

(iii) Let X be a subset of $\mathbb{R}^n$, and let a be a point in X. What does it mean to say that X is *star-shaped* around a? Show that if X is star-shaped around a, then it is path-connected.

(iv) Suppose that $f: X \to \mathbb{R}$ is continuous, $f(x)$ is nonzero for all x, and there exist $x_0, x_1 \in X$ with $f(x_0) < 0 < f(x_1)$. Prove that X is not path-connected.

(v) Recall that $GL_3(\mathbb{R})$ is the space of 3×3 invertible matrices over $\mathbb{R}$. Prove that this space is not path-connected.

2 (i) Let X be a metric space, and let x_0 and x_1 be points in X. What does it mean to say that two paths from x_0 to x_1 are *homotopic relative to endpoints*? Define the set $\pi_1(X; x_0, x_1)$.

(ii) Let X be path-connected. Prove that the group $\pi_1(X; x_0)$ is isomorphic to the group $\pi_1(X; x_1)$.

(iii) Put $X = \{(w, x, y, z) \in \mathbb{C}^4 \mid w \neq x \ , \ x \neq y \ , \ y \neq z\}$, and take $x_0 = (0, 1, 2, 3)$ as the basepoint in X. Calculate $\pi_1 X$. (You may wish to consider the expression $f(w, x, y, z) = (w, x - w, y - x, z - y)$.)

3 Are the following statements true or false? Give proof or disproof as appropriate. You may quote general theorems without proof, provided that you state them clearly. You may give pictures instead of formulae, provided that they are clearly explained.

(i) There is a continuous surjective map from $S^1 \times S^1$ to $\mathbb{R}$

(ii) $\mathbb{C} \setminus \{2\}$ is homotopy equivalent to S^1

(iii) $\mathbb{C} \setminus \{-1, 1\}$ is homotopy equivalent to S^1

(iv) $S^2 \setminus \{\text{the north pole}\}$ is homeomorphic to $\mathbb{C}$.

(v) The letter X (considered as a subspace of $\mathbb{R}^2$) is homeomorphic to the letter Y.

(vi) The letter X (considered as a subspace of $\mathbb{R}^2$) is homotopy equivalent to the letter Y.

4 Give examples of the following things, with justification.

(i) Connected sets $X, Y \subseteq \mathbb{R}^2$ such that $X \cap Y$ is not connected.

(ii) A sequence of open sets $U_n \subseteq \mathbb{R}$ such that the set $X = U_1 \cap U_2 \cap \ldots = \bigcap_n U_n$ is not open.

(iii) A surjective map $f: X \to Y$ of based spaces such that the homomorphism $f_*: \pi_1 X \to \pi_1 Y$ is not surjective.

(iv) A space X that is homotopy equivalent to $X \times X$.

(v) A space X that is not homotopy equivalent to $X \times X$.

5 (i) Let X be a based metric space, and let Y be a subspace of X containing the basepoint. What does it mean to say that Y is a *retract* of X?

(ii) Prove that if Y is a retract of X, then $|\pi_1 Y| \leqslant |\pi_1 X|$.

(iii) Recall that $\mathbb{R}P^3$ is a subspace of the space $M_4\mathbb{R}$ of all 4×4 matrices over $\mathbb{R}$, which is homeomorphic to $\mathbb{R}^{16}$. Prove that $\mathbb{R}P^3$ is not a retract of $M_4\mathbb{R}$.

(iv) Recall that U_3 is the space of 3×3 matrices A over $\mathbb{C}$ such that $A^\dagger A = I$. You may assume that for such A we have $\det(A) \in S^1$. Define $j: S^1 \to U_3$ by

$$j(z) = \begin{pmatrix} z & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

What is $\det(j(z))$? Deduce that $\pi_1 U_3$ is infinite.

End of Question Paper