ALGEBRAIC TOPOLOGY SAMPLE EXAM — SOLUTIONS

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1. (a) A space X is *path-connected* if for each pair of points $x_0, x_1 \in X$ there exists a continuous map $u\colon I \to X$ such that $u(0) = x_0$ and $u(1) = x_1$.
   (b) Suppose that $n > 1$ and that $x_0, x_1 \in S^n$. Suppose first that $x_1 \neq -x_0$, so that the line segment from x_0 to x_1 does not pass through the origin. Thus, if we put $f(t) = (1-t)x_0 + tx_1$ then $f(t) \neq 0$ for all $t \in I$. We can thus define a continuous map $u\colon I \to S^n$ by $u(t) = f(t)/\|f(t)\|$ and this satisfies $u(0) = x_0/\|x_0\| = x_0$ and $u(1) = x_1$ as required.
   Now consider the exceptional case where $x_1 = -x_0$. As $n > 0$ the set S^n has more than two points so we can choose a point x_2 that is different from both $-x_0$ and $-x_1$. By the first part of the proof we can define a path u from x_0 to x_2 and a path v from x_1 to x_2 in S^n. This gives a path $w := u * \overline{v}$ from x_0 to x_2.
   (c) A subset $X \subseteq \mathbb{R}^n$ is *star-shaped* around a point $a \in X$ if for all $x \in X$, the linear path from x to a (given by the formula $u(t) = (1-t)x + ta$, which is meaningful because x and a are vectors in $\mathbb{R}^n$) lies wholly in X.
   Suppose that this holds. For any $x_0, x_1 \in X$ we can let u_0 be the linear path from x_0 to a and let u_1 be the linear path from x_1 to a. Then $u_0 * \overline{u}_1$ is a path from x_0 to x_1, showing that X is path-connected.
   (d) Suppose that $f\colon X \to \mathbb{R}$ is continuous, $f(x)$ is nonzero for all x, and there exist $x_0, x_1 \in X$ with $f(x_0) < 0 < f(x_1)$. I claim that there is no continuous path in X from x_0 to x_1, so that X is not path-connected. Indeed, if u is such a path, put $g(t) = f(u(t))$, giving a continuous function $g\colon I \to \mathbb{R}$. We have $g(0) = f(x_0) < 0$ and $g(1) = f(x_1) > 0$. By the Intermediate Value Theorem, there must be some $t \in I$ with $g(t) = 0$, or in other words $f(u(t)) = 0$. However, $u(t) \in X$, and $f(x) \neq 0$ for all $x \in X$ by assumption. This contradiction shows that there can be no such map u.
   (e) Consider the map $\det\colon GL_3(\mathbb{R}) \to \mathbb{R}$. As $\det(A)$ is a polynomial expression in the entries of the matrix A, we see that det is continuous. If $A \in GL_3(\mathbb{R})$ then A is invertible, so $\det(A) \neq 0$. The matrices

$$A_0 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

   lie in $GL_3(\mathbb{R})$ and satisfy $\det(A_0) < 0 < \det(A_1)$. It follows from the previous part that $GL_3(\mathbb{R})$ is disconnected.
2. (a) Let X be a metric space, and let x_0 and x_1 be two points in X. Let $u, v\colon I \to X$ be two paths, both of which start at x_0 and end at x_1. We say that u and v are *homotopic relative to endpoints* if there exists a map $h\colon I \times I \to X$ such that
   - $h(0,t) = u(t)$ for all $t \in I$
   - $h(1,t) = v(t)$ for all $t \in I$
   - $h(s,0) = x_0$ for all $s \in I$
   - $h(s,1) = x_1$ for all $s \in I$.

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This is an equivalence relation on the set of all paths from x_0 to x_1 in X; the set $\pi_1(X;x_0,x_1)$ is just the set of equivalence classes.

(b) Now suppose that X is path-connected, so we can choose a path u from x_0 to x_1 in X, and put $q=[u]\in\pi_1(X;x_0,x_1)$. If $a\in\pi_1(X;x_0)=\pi_1(X;x_0,x_0)$ then q^{-1} runs from x_1 to x_0, and a runs from x_0 to x_0, and q runs from x_0 to x_1, so $q^{-1}aq$ runs from x_1 to itself. We can thus define a function $f\colon\pi_1(X;x_0)\to\pi_1(X;x_1)$ by $f(a)=q^{-1}aq$. Similarly, we can define $g\colon\pi_1(X;x_1)\to\pi_1(X;x_0)$ by $g(b)=qbq^{-1}$. Clearly $g(f(a))=qq^{-1}aqq^{-1}=a$ and similarly $f(g(b))=b$, so f is a bijection with inverse g. Moreover, $f(a)f(a')=q^{-1}aqq^{-1}a'q=q^{-1}aa'q=f(aa')$, so f is a group homomorphism, and thus an isomorphism $\pi_1(X;x_0)\to\pi_1(X;x_1)$.

(c) Define $f\colon X\to\mathbb{C}\times\mathbb{C}^\times\times\mathbb{C}^\times\times\mathbb{C}^\times$ by $f(w,x,y,z)=(w,x-w,y-x,z-y)$ (where $\mathbb{C}^\times$ means $\mathbb{C}\setminus\{0\}$). This is a homeomorphism, with inverse $f^{-1}(a,b,c,d)=(a,a+b,a+b+c,a+b+c+d)$. On the other hand, $\mathbb{C}$ is homotopy equivalent to a point, and $\mathbb{C}^\times$ is homotopy equivalent to S^1; it follows that X is homotopy equivalent to $S^1\times S^1\times S^1$, and thus that π_1X is isomorphic to $\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}$.

3. (a) **False.** The space $S^1\times S^1$ is compact, and $\mathbb{R}$ is not compact, so there can be no continuous surjection from $S^1\times S^1$ to $\mathbb{R}$.

(b) **True.** The map $f\colon\mathbb{C}\setminus\{2\}\to S^1$ given by $f(z)=(z-2)/|z-2|$ is a homotopy equivalence.

(c) **False.** The space $\mathbb{C}\setminus\{-1,1\}$ is homotopy equivalent to the figure eight, so its fundamental group is nonabelian, whereas π_1S^1 is isomorphic to $\mathbb{Z}$ and thus is abelian. This shows that the two spaces have non-isomorphic fundamental groups, so they cannot be homotopy equivalent.

(d) **True.** The space $S^2\setminus\{\text{the north pole}\}$ is homeomorphic to $\mathbb{R}^2$ by stereographic projection, and of course $\mathbb{R}^2$ is homeomorphic to $\mathbb{C}$ by the correspondence $(x,y)\leftrightarrow x+iy$.

(e) **False.** We can remove the central point from the letter X and the resulting space has four path components; but if we remove a point from the letter Y, the remaining space has at most three path components. This shows that X is not homeomorphic to Y.

(f) **True.** The letter X is star-shaped around its central point, so it is contractible, and the same applies to Y. Thus, they are both homotopy equivalent to a point and hence to each other.

4. (a) Put

$$\begin{aligned} X &= \text{the upper half of the unit circle} \\ &= \{(x,y)\in\mathbb{R}^2 \mid x^2+y^2=1\ ,\ y\geq 0\} \\ Y &= \text{the lower half of the unit circle} \\ &= \{(x,y)\in\mathbb{R}^2 \mid x^2+y^2=1\ ,\ y\leq 0\} \end{aligned}$$

Then X and Y are both connected, but $X\cap Y=\{(-1,0),(1,0)\}$, which is disconnected.

(b) Put $U_n=\{x\in\mathbb{R}\mid |x|<1/n\}=(-1/n,1/n)$. Then U_n is open in $\mathbb{R}$, but

$$\begin{aligned} \bigcap_n U_n &= \{x\in\mathbb{R}\mid |x|<1/n \text{ for all } n\} \\ &= \{x\in\mathbb{R}\mid |x|=0\} \\ &= \{0\}, \end{aligned}$$

which is not open.

(c) Define $\eta\colon\mathbb{R}\to S^1$ by $\eta(t)=\exp(2\pi it)$, which gives a surjective, continuous map. As $\pi_1\mathbb{R}=\{e\}$ and π_1S^1 is infinite, it is clear that $\eta_*\colon\pi_1\mathbb{R}\to\pi_1S^1$ cannot be surjective.

(d) The spaces I and $I\times I$ are both homotopy equivalent to a point, and thus to each other.

(e) The space S^1 is not homotopy equivalent to $S^1\times S^1$ (because $\pi_1S^1=\mathbb{Z}$ is not isomorphic to $\pi_1(S^1\times S^1)=\mathbb{Z}\times\mathbb{Z}$).

5. (a) Let X be a based metric space, let Y be a subspace of X containing the basepoint, and let $i\colon Y\to X$ be the inclusion map. We say that Y is a *retract* of X if there exists a continuous map $r\colon X\to Y$ such that $r\circ i=1_Y$, or equivalently $r(y)=y$ for all $y\in Y$.

(b) Suppose that Y is a retract of X. We then have homomorphisms $i_*\colon \pi_1 Y \to \pi_1 X$ and $r_*\colon \pi_1 X \to \pi_1 Y$ such that $r_* i_* = 1\colon \pi_1 Y \to \pi_1 Y$. Now let a and a' be two different elements of $\pi_1 Y$. Then $r_*(i_*(a)) = a \neq a' = r_*(i_*(a'))$, so clearly $i_*(a)$ cannot be the same as $i_*(a')$. Thus, all the different elements of $\pi_1 Y$ are mapped to different elements of $\pi_1 X$, so there must be at least as many elements in $\pi_1 X$ as there are in $\pi_1 Y$. In other words, we have $|\pi_1 Y| \leq |\pi_1 X|$.

(c) We have $|\pi_1 \mathbb{R}P^3| = 2$ and $M_4\mathbb{R} \simeq \mathbb{R}^{16}$ is contractible so $|\pi_1 M_4\mathbb{R}| = 1$. By the previous part, $\mathbb{R}P^3$ cannot be a retract of $M_4\mathbb{R}$.

(d) It is easy to see that $\det(j(z)) = z$, so $\det \circ j = 1\colon S^1 \to S^1$. It follows that the maps $j_*\colon \pi_1 S^1 \to \pi_1 U_3$ and $\det_*\colon \pi_1 U_3 \to \pi_1 S^1$ satisfy $\det_* \circ j_* = 1\colon \pi_1 S^1 \to \pi_1 S^1$. By the logic of part (ii) we see that $|\pi_1 U_3| \geq |\pi_1 S^1| = |\mathbb{Z}| = \infty$.