PURE MATHEMATICS

Spring Semester 2001 $2\frac{1}{2}$ **hours**

Algebraic Topology — Past Exam Questions

This "paper" consists of a collection of past exam questions, often modified to make them compatible with the way the course was taught this year. As a guide to the nature of the actual exam, they are not as good as the sample exam that was distributed separately, but they may be useful nonetheless.

1 Let X be a metric space.

(i) Let Y be a compact subspace of X. Prove that Y is closed in X.

(ii) Let Y and Z be two compact subspaces of X. Prove that $Y \cup Z$ is compact.

(iii) Deduce (or prove otherwise) that every finite space is compact.

(iv) Let Y and Z be compact metric spaces. Prove that $Y \times Z$ is compact.

(v) Conversely, let Y and Z be metric spaces such that $Z \neq \emptyset$ and $Y \times Z$ is compact. Prove that Y is compact.

(vi) Put $X = \{(x, y, z) \in \mathbb{R}^3 \ : \ x^4 + y^4 + z^4 = 1\}$. Prove that X is compact. You may use general theorems provided that you state them precisely.

2 (i) Let X be a metric space. Define the equivalence relation $\sim$ on X such that $\pi_0(X) = X/\sim$, and prove that it is an equivalence relation.

(ii) Let $f\colon X \longrightarrow Y$ be a continuous map. Define the induced map $f_*\colon \pi_0 X \longrightarrow \pi_0 Y$, and prove that it is well-defined.

(iii) Show that if $f, g\colon X \longrightarrow Y$ are homotopic maps then $f_* = g_*\colon \pi_0 X \longrightarrow \pi_0 Y$.

(iv) Put $X = [-3, -2] \cup [-1, 1] \cup [2, 3]$ and $Y = [0, 1] \cup [2, 10]$, and define $f\colon X \longrightarrow Y$ by $f(x) = x^2$. Describe the sets $\pi_0 X$ and $\pi_0 Y$ and the map $f_*\colon \pi_0 X \longrightarrow \pi_0 Y$.

3 Are the following statements true or false? Give proof or disproof as appropriate. You may quote general theorems or calculations of homology groups without proof, provided that you state them clearly. You may give pictures instead of formulae, provided that they are clearly explained.

(i) $\mathbb{R}P^1$ is homeomorphic to S^1.

(ii) The Möbius strip is homotopy equivalent to S^2.

(iii) $S^2 \setminus S^1$ is homotopy equivalent to $\mathbb{R} \setminus \{0\}$.

(iv) The letter A is homeomorphic to the letter D.

(v) Any compact convex subset of $\mathbb{R}^2$ is homeomorphic to B^2.

4 Give examples of the following things, with careful justification.

(i) A noncompact metric space X with a sequence of compact subspaces $Y_1 \subset Y_2 \subset \ldots$ such that the union of all the sets Y_n is equal to X.

(ii) A metric space X with two noncompact subsets Y, Z such that $Y \cup Z$ is compact.

(iii) A sequence in $\mathbb{R}$ with no convergent subsequence.

(iv) A non-surjective map $f: X \longrightarrow Y$ such that $f_*: \pi_0 X \longrightarrow \pi_0 Y$ is surjective.

(v) An injective map $f: X \longrightarrow Y$ such that $f_*: \pi_0 X \longrightarrow \pi_0 Y$ is not injective.

5 Are the following statements true or false? Give proof or disproof as appropriate. You may quote general theorems, provided that you state them clearly.

(i) The punctured disc $X = \{(x, y) \in \mathbb{R}^2 \mid 0 < x^2+y^2 \leqslant 1\}$ is compact.

(ii) The circle S^1 is homeomorphic to $S^1 \times I$.

(iii) The circle S^1 is homotopy equivalent to $S^1 \times I$.

(iv) $\mathbb{C} \setminus S^1$ is homotopy equivalent to $Y = \{z \in \mathbb{C} \mid z = 0 \text{ or } |z| = 1\}$.

(v) Every continuous bijection from $[0, 1] \cup (2, 3]$ to $[0, 1]$ is a homeomorphism.

6 (i) What is a *metric space*? What is a *continuous function*?

(ii) Define the discrete metric on a set X.

(iii) Let X be a space with a discrete metric. Show that any path $s\colon \Delta_1 \longrightarrow X$ is constant, and deduce that $\pi_0 X = X$.

(iv) Consider the space $Y = \{(x, y) \in \mathbb{R}^2 \mid xy \neq 0\}$ and show that $\pi_0 Y$ has precisely four elements. If $f\colon Y \longrightarrow Y$ denotes reflection in the line $x = y$, describe the map $f_*\colon \pi_0 Y \longrightarrow \pi_0 Y$. If f homotopic to the identity map?

7 Are the following statements true or false? Give proof or disproof as appropriate. You may quote general theorems, provided that you state them clearly.

(i) S^1 is contractible.

(ii) If a space X is the union of two closed, path-connected subspaces A and B, then X is path-connected.

(iii) $(\mathbb{R} \times \mathbb{R}) \setminus (\mathbb{R} \times \{0\})$ is homotopy equivalent to S^1.

(iv) $(\mathbb{R} \times \mathbb{R}^2) \setminus (\mathbb{R} \times \{0\})$ is homotopy equivalent to S^1.

(v) The space $\mathbb{C} \setminus \{0, 1\}$ is homeomorphic to $\mathbb{C} \setminus \{i, -i\}$.

(vi) The space $\mathbb{C} \setminus \{0, 1\}$ is homotopy equivalent to $\mathbb{C} \setminus \{0, 1, 2\}$.

8 Are the following statements true or false? Give proof or disproof as appropriate. You may quote general theorems, provided that you state them clearly.

(i) The identity map of the unit circle is homotopic to the constant map $c\colon S^1 \longrightarrow S^1$ defined by $c(z) = 1$ for all z.

(ii) Let $f_n\colon S^1 \longrightarrow S^1$ be defined by $f_n(z) = z^n$. Then f_n is not homotopic to f_m when $n \neq m$.

(iii) $\mathbb{R}^2$ is homeomorphic to $\mathbb{R}^3$.

(iv) If $f\colon X \longrightarrow X$ is a homotopy equivalence, then $f_*\colon \pi_1 X \longrightarrow \pi_1 X$ is the identity map.

9 (i) Let $f, g\colon X \longrightarrow Y$ be continuous maps between metric spaces. What does it mean to say that f is homotopic to g?

(ii) Let X and Y be metric spaces. What does it mean to say that X and Y are homotopy equivalent?

(iii) Show that if X and Y are homotopy equivalent then there is a bijection between the sets of path-components $\pi_0 X$ and $\pi_0 Y$.

(iv) Consider the cross $X = \{(x, 0) \mid -1 \leqslant x \leqslant 1\} \cup \{(0, y) \mid -1 \leqslant y \leqslant 1\}$, and let $C = \mathbb{R}^2 \setminus X$ be its complement. Prove that C is homotopy equivalent to S^1.

10 Consider a metric space X.

(i) (a) What does it mean to say that a subset U of X is open?

(b) What does it mean to say that a subset F of X is closed?

(c) Show that a subset $F \subseteq X$ is closed iff for every sequence (x_n) in F that converges to a point $x \in X$, we actually have $x \in F$.

(ii) Explain what it means for a subset $A \subseteq X$ to be compact. Show that if A is compact and $f\colon X \longrightarrow Y$ is continuous then $f(A)$ is compact.

(iii) Prove that the space $[0, 1]$ is compact. Show that there is a continuous bijection $g\colon [-1, -1/2) \cup [1/2, 1] \longrightarrow [0, 1]$; can it be chosen to be a homeomorphism?

11 Are the following statements true or false? Give proof or disproof as appropriate. You may quote general theorems, provided that you state them clearly.

(i) The torus $T = S^1 \times S^1$ is homotopy equivalent to S^2.

(ii) There is a map $r\colon B^2 \longrightarrow S^1$ such that rj is homotopic to 1_{S^1}, where $j\colon S^1 \longrightarrow B^2$ is the inclusion map.

(iii) $\mathbb{R}^2$ is homeomorphic to $\mathbb{R}^3$.

(iv) Every continuous function $f\colon S^2 \longrightarrow \mathbb{R}^3$ is homotopic to a constant function.

12 (i) What does it mean to say that a metric space X is *homotopy equivalent* to a metric space Y? Show that the relation of homotopy equivalence is an equivalence relation.

(ii) What does it mean for a space to be (a) *contractible* and (b) *path connected*? Show that any contractible space is path connected. Is the reverse implication true?

(iii) Consider the rational comb space

$$X = \{(x, y) \in \mathbb{R}^2 \mid y \geqslant 0 \text{ or } x \in \mathbb{Q}\}.$$

Show that X is homotopy equivalent to the upper half plane $Y = \{(x, y) \in \mathbb{R}^2 \mid y \geqslant 0\}$, and deduce that X is contractible.

End of Question Paper