Algebraic Topology Past Exam Questions — Solutions

1. (i) Let (y_n) be a sequence in Y, converging to some point $x \in X$. Clearly any subsequence converges to x also. By compactness, some subsequence (y_{n_k}) converges to some $y \in Y$, and as limits are unique we must have $x = y$, so $x \in Y$. This means that Y is closed, as required.

   (ii) Let (x_n) be a sequence in $Y \cup Z$. Then either $x_n \in Y$ for infinitely many n, or $x_n \in Z$ for infinitely many n. In the first case, we can choose a subsequence (x'_n) of (x_n) such that $x'_n \in Y$ for all n in other words we have a sequence in Y. As Y is compact, some subsequence (x''_n) of (x'_n) converges in Y, and thus in $Y \cup Z$. The other case is similar, so in either case some subsequence of (x_n) converges in $Y \cup Z$. This implies that $Y \cup Z$ is compact.

   (iii) If X has only one point then every sequence converges so X is compact. If X has $n > 1$ points, we can write it in the form $X = Y \cup Z$ where $|Y| = n - 1$ and $|Z| = 1$, so Y and Z are compact by induction, so X is compact by (ii).

   (iv) Let (w_n) be a sequence in $Y \times Z$, with $w_n = (y_n, z_n)$ say. As Y is compact, some subsequence (y_{n_k}) converges to some $y \in Y$. Put $y'_k = y_{n_k}$ and $z'_k = z_{n_k}$ and $w'_k = (y'_k, z'_k) = w_{n_k}$. As Z is compact, some subsequence z'_{k_j} converges to some point $z \in Z$. Put $y''_j = y'_{k_j}$ and $z''_j = z'_{k_j}$ and $w''_j = (y''_j, z''_j) = w'_{k_j}$. As (y''_j) is a subsequence of the sequence (y'_k) which converges to y, we see that $y''_j \to y$. By assumption we have $z''_j \to z$, so $w''_j \to (y, z)$. Thus, some subsequence of (w_n) converges in $Y \times Z$, proving that $Y \times Z$ is compact as claimed.

   (v) As $Z \neq \emptyset$ we can choose a point $a \in Z$. Let $p\colon Y \times Z \to Y$ be defined by $p(y, z) = y$. We have $p(y, a) = y$, which shows that p is surjective. In general, if $f\colon A \to B$ is a surjective continuous map of spaces and A is compact we know that B is compact. As $Y \times Z$ is assumed compact, we deduce that Y is compact.

   (vi) If $(x, y, z) \in X$ then $x^4 \leq x^4 + y^4 + z^4 = 1$ so $|x| \leq 1$. Similarly, we see that $|y| \leq 1$ and $|z| \leq 1$, which implies that X is bounded. I claim that it is also closed in $\mathbb{R}^3$. Indeed, suppose we have a sequence $a_n = (x_n, y_n, z_n)$ in X converging to some point $a = (x, y, z) \in \mathbb{R}^3$. then $x_n^4 + y_n^4 + z_n^4 = 1$ and $x_n \to x$, $y_n \to y$ and $z_n \to z$, so by the algebra of limits we have

   $$x^4 + y^4 + z^4 = \lim(x_n^4 + y_n^4 + z_n^4) = 1,$$

   so $a \in X$.

   A bounded closed subset of $\mathbb{R}^n$ is compact, so we deduce that X is compact as claimed.

2. (i) We write $x \sim y$ iff there is a path in X from x to y, in other words a continuous map $s\colon I \to X$ such that $s(0) = x$ and $s(1) = y$. For any $x \in X$ we can define $c_x\colon I \to X$ by $c_x(t) = x$ for all t; this is a path from x to x, proving that $x \sim x$. If $x \sim y$ then there is a path s from x to y and we can define a path $\overline{s}$ from y to x by $\overline{s}(t) = s(1 - t)$; this shows that $y \sim x$. If there is also a path r from y to z then we can define a path $s * r$ from x to z by

   $$(s * r)(t) = \begin{cases} s(2t) & \text{if } 0 \leq t \leq 1/2 \\ r(2t - 1) & \text{if } 1/2 \leq t \leq 1, \end{cases}$$

   and this shows that $x \sim z$. Thus $\sim$ is reflexive, symmetric and transitive and thus is an equivalence relation.

   (ii) Let c be an element of $\pi_0 X$, in other words a path component in X. For any $x \in c$ we have a point $f(x) \in Y$, and thus a path-component $\langle f(x) \rangle \in \pi_0 Y$. If x' is another point in c then $x \sim x'$ so we can choose a path s from x to x' in X. Thus $f \circ s\colon I \to Y$ is a path in Y from $f(x)$ to $f(x')$, so $f(x) \sim f(x')$, so $\langle f(x) \rangle = \langle f(x') \rangle$. We can thus define $f_*(c) = \langle f(x) \rangle$; this is independent of the choice of x and thus is well-defined.

   (iii) If $f, g\colon X \to Y$ are homotopic then we can chooose a map $h\colon I \to X \to Y$ such that $h(0, x) = f(x)$ and $h(1, x) = g(x)$ for all x. If $c \in \pi_0 X$ we can choose $x \in X$ and note taht $f_*(c) = \langle f(x) \rangle$ and $g_*(c) = \langle g(x) \rangle$. We can also define a map $s\colon I \to Y$ by

$s(t) = h(t,x)$. This gives a path from $s(0) = f(x)$ to $s(1) = g(x)$, so $\langle f(x)\rangle = \langle g(x)\rangle$, in other words $f_*(c) = g_*(c)$.

(iv) Write

$$\begin{aligned} a &= [-3,-2] \\ b &= [-1,1] \\ c &= [2,3] \\ d &= [0,1] \\ e &= [2,11] \end{aligned}$$

Then $\pi_0 X = \{a,b,c\}$ and $\pi_0 Y = \{c,d\}$. The map $f_*\colon \pi_0 X \to \pi_0 Y$ is given by $f_*(a) = f_*(c) = e$ and $f_*(b) = d$.

3. (i) True. There is a homeomorphism $f\colon S^1 \to \mathbb{R}P^1$ given by

$$f(x,y) = \frac{1}{2}\begin{pmatrix} 1+x & y \\ y & 1-x \end{pmatrix}.$$

(ii) This is false, because the Möbius strip M is homotopy equivalent to S^1, so $\pi_1 M \simeq \pi_1 S^1 \simeq \mathbb{Z}$, whereas $\pi_1 S^2 = 0$.

(iii) This is true because both spaces are homotopy equivalent to the space with two points. Indeed, $\mathbb{R}\setminus\{0\}$ is the disjoint union of two contractible spaces $(-\infty,0)$ and $(0,\infty)$, each of which is homotopy equivalent to a point, so $\mathbb{R}\setminus\{0\}$ is homotopy equivalent to two points. Similarly, $S^2\setminus S^1$ is the disjoint union of the sets $U_+ = \{(x,y,z)\in S^2 \mid z>0\}$ and $U_- = \{(x,y,z)\in S^2 \mid z<0\}$. If we put $V = \{(x,y)\in\mathbb{R}^2 \mid x^2+y^2<1\}$ then V is contractible and there is a homeomorphism $f_+\colon V\to U_+$ given by $f(x,y) = (x,y,\sqrt{1-x^2-y^2})$, so U_+ is contractible. Similarly U_- is contractible, so $S^2\setminus S^1$ is again homotopy equivalent to two points.

(iv) This is false, because A can be disconnected by removing a point in the middle of one of the legs, but D cannot be disconnected by removing a single point.

(v) This is false: the closed line segment from $(-1,0)$ to $(1,0)$ is compact and convex but not homeomorphic to B^2. (The theorem states that if $X\subseteq\mathbb{R}^2$ is compact and convex *and contains an open ball* then X is homeomorphic to B^2.)

4. (i) Take $X=\mathbb{R}$, $Y_n=[-n,n]$. The sequence $(1,2,3,\dots)$ in $\mathbb{R}$ has no convergent subsequence, so $\mathbb{R}$ is noncompact. Moreover, Y_n is a bounded closed subspace of $\mathbb{R}$ and thus is compact. For any $x\in\mathbb{R}$ we can choose an integer $n>|x|$ and then $x\in Y_n$, which shows that $X = Y_1\cup Y_2\cup\dots$.

(ii) Put $X=[0,1]$ and $Y=(0,1]$ and $Z=[0,1)$. Then $Y\cup Z=[0,1]$ which is compact. The sequence $(1/n)$ in Y has n subsequence that converges in Y, so Y is noncompact. Similarly, the sequence $(1-1/n)$ in Z has no subsequence that converges in Z, so Z is noncompact.

(iii) The sequence $1,2,3,\dots$ in $\mathbb{R}$ has no convergent subsequence, because any two distict terms have distance at least one apart so no subsequence can be Cauchy.

(iv) Let $X=\{0\}$ and $Y=[0,1]$ and define $f\colon X\to Y$ by $f(0)=0$. Then X and Y are both path-connected, so π_0X and π_0Y have only one point each. The map $f_*\colon\pi_0X\to\pi_0Y$ sends the only point in π_0X to the only point in π_0Y, so f_* is a bijection and in particular is surjective. However f is obviously not surjective, as 1 does not lie in the image of f for example.

(v) Put $X=\{0,1\}$ and $Y=[0,1]$ and let $f\colon X\to Y$ be the inclusion map, which is clearly injective. If we write a for the component of 0 in X and b for the component of 1 in X and c for the component of 0 in Y then $\pi_0X=\{a,b\}$ and $\pi_0Y=\{c\}$ and $f_*(a)=f_*(b)=c$, so f_* is not injective.

5. (i) False. The space X is not closed in $\mathbb{R}^2$, because the sequence $(0,1/n)$ in X converges in $\mathbb{R}^2$ to the point $(0,0)$, which does not lie in X. A subspace of $\mathbb{R}^n$ is compact iff it is bounded and closed, so X is not compact.

(ii) False. Removing any two points disconnects S^1, but $S^1 \times I$ cannot be disconnected by removing any finite set of points.

(iii) True. Define maps as follows:

$$\begin{aligned} &f\colon S^1 \to S^1 \times I && f(z) = (z,0) \\ &g\colon S^1 \times I \to S^1 && g(z,r) = z \\ &h\colon I \times (S^1 \times I) \to S^1 \times I && h(t,(z,r)) = (z,tr). \end{aligned}$$

Then $gf = 1\colon S^1 \to S^1$ and h is a (linear) homotopy from fg to $1_{S^1 \times I}$, so f is a homotopy equivalence with homotopy inverse g.

(iv) True. Define maps as follows:

$$\begin{aligned} &f\colon \mathbb{C} \setminus S^1 \to Y && f(z) = \begin{cases} z/|z| & \text{if } |z| > 1 \\ 0 & \text{if } |z| < 1 \end{cases} \\ &g\colon Y \to \mathbb{C} \setminus S^1 && g(z) = 2z. \end{aligned}$$

Then $fg = 1_Y$ and gf is linearly homotopic to $1_{\mathbb{C}\setminus S^1}$, so f is a homotopy equivalence with homotopy inverse g.

(v) False. Define $f\colon [0,1] \cup (2,3] \to [0,1]$ by

$$f(t) = \begin{cases} t/2 & \text{if } t \in [0,1] \\ (t-1)/2 & \text{if } t \in (2,3]. \end{cases}$$

Then f is a continuous bijection, but f^{-1} is not continuous (because $1/2 + 1/2n \to 1/2$ but $f^{-1}(1/2 + 1/2n) = 2 + 1/n$ does not converge to $f^{-1}(1/2) = 1$), so f is not a homeomorphism.

6. (i) A metric space is a set X equipped with a metric, ie a function $d\colon X \times X \to \mathbb{R}$ such that

- $d(x,y) \geq 0$ for all $x, y \in X$, with equality iff $x = y$.
- $d(x,y) = d(y,x)$ for all $x, y \in X$.
- $d(x,z) \leq d(x,y) + d(y,z)$ for all $x, y, z \in X$.

A function $f\colon X \to Y$ between metric spaces is continuous if for each sequence (x_n) in X that converges to a point $x \in X$, the resulting sequence $(f(x_n))$ in Y converges to the point $f(x)$.

(ii) The discrete metric on a set X is defined by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y. \end{cases}$$

(iii) Let $s\colon I \to X$ be a path. Define $f\colon I \to \mathbb{R}$ by $f(t) = d(s(t), s(0))$, so f is continuous and $f(0) = 0$. As d can only take the values 0 and 1, we see that f can only take the vales 0 and 1, so by the intermediate value theorem it must be constant. As $f(0) = 0$ we see that $f(t) = 0$ for all t. As $d(s(0), s(t)) = 0$ we see that $s(t) = s(0)$ for all t, in other words s is constant.

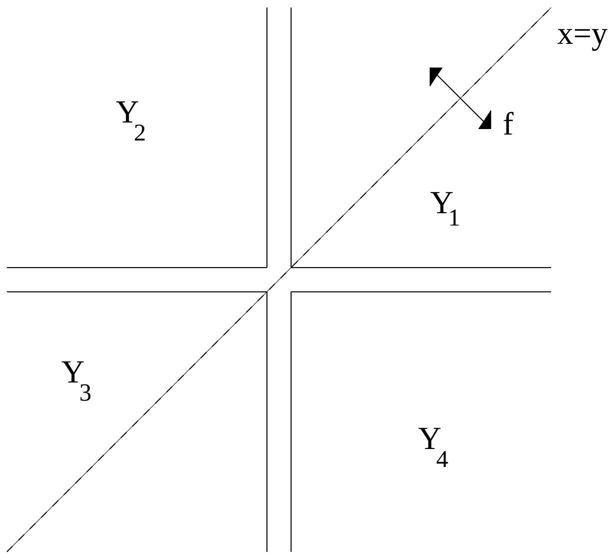
As usual we write $x \sim y$ if x can be connected to y by a path, so $\sim$ is an equivalence relation and $\pi_0 X = X/\sim$. As the only paths are constant, if $x \sim y$ we must have $x = y$. Thus, each equivalence class consists of just a single point, so $\pi_0 X$ can be identified with X.

(iii) Define

$$\begin{aligned} Y_1 &= \text{ 1st quadrant } = \{(x,y) \mid x > 0, y > 0\} \\ Y_2 &= \text{ 2nd quadrant } = \{(x,y) \mid x < 0, y > 0\} \\ Y_3 &= \text{ 3rd quadrant } = \{(x,y) \mid x < 0, y < 0\} \\ Y_4 &= \text{ 4th quadrant } = \{(x,y) \mid x > 0, y < 0\}. \end{aligned}$$

These are all nonempty convex sets and thus path-connected, and clearly they are open and disjoint. If $(x,y) \in Y$ then $xy \neq 0$ so $x \neq 0$ and $y \neq 0$ so $x < 0$ or $x > 0$ and $y < 0$ or $y > 0$. It follows that (x,y) lies in one of the sets Y_i, so $Y = Y_1 \cup Y_2 \cup Y_3 \cup Y_4$. A path in Y has the form $s(t) = (u(t), v(t))$, where for all t we have $u(t) \neq 0$ and $v(t) \neq 0$. By the intermediate value theorem, we see that $u(0)$ has the same sign as $u(1)$, and $v(0)$ has the same sign as $v(1)$, so if $s(0) \in Y_i$ then $s(1) \in Y_i$ also. It follows that the sets Y_i are the path components of Y, so $\pi_0 Y = \{Y_1, Y_2, Y_3, Y_4\}$. The formula for the map f is $f(x, y) = (y, x)$, and it follows easily that

$$f_*(Y_1) = Y_1 \qquad f_*(Y_2) = Y_4 \qquad f_*(Y_3) = Y_3 \qquad f_*(Y_4) = Y_2.$$



As f_* is not the identity map, we see that f is not homotopic to the identity.

7. (i) False. We have $\pi_1 S^1 = \mathbb{Z}$ but π_1 of a point is the trivial group, so S^1 is not homotopy equivalent to a point.

(ii) False. Put $X = \{0, 1\}$ and $A = \{0\}$ and $B = \{1\}$. Then A and B are closed path connected subsets of X with $X = A \cup B$, but X is not path connected. (You would not be required to say this, but I remark that if $X = A \cup B$ where A and B are path connected (not necessarily closed) and $A \cap B \neq \emptyset$ then X is path connected.)

(iii) False. Write

$$X = (\mathbb{R} \times \mathbb{R}) \setminus (\mathbb{R} \times \{0\}) = \{(x, y) \in \mathbb{R}^2 \mid y \neq 0\}.$$

We can then define a map $f : X \to \mathbb{R}$ by $f(x, y) = y$. This is never zero and it is positive at $(0, 1)$ and negative at $(0, -1)$, so $(0, 1)$ cannot be joined to $(0, -1)$ by a path in X, so X is not path connected. However, S^1 is path connected and anything homotopy equivalent to a path connected space is again path connected so X is not homotopy equivalent to S^1.

(iv) True. Write $Y = (\mathbb{R} \times \mathbb{R}^2) \setminus (\mathbb{R} \times \{0\})$, and define maps as follows

$$f : Y \to S^1 \qquad f(x, y, z) = (y, z)/\sqrt{y^2 + z^2}$$
$$g : S^1 \to Y \qquad g(y, z) = (0, y, z).$$

Then $gf = 1_{S^1}$, and fg is linearly homotopic to 1_Y.

(v) True. We can define a homeomorphism $f : \mathbb{C} \setminus \{0, 1\} \to \mathbb{C} \setminus \{i, -i\}$ by $f(z) = 2iz - i$, with inverse $f^{-1}(w) = (w + i)/2i$.

(vi) False. We have $H_1(\mathbb{C} \setminus \{0, 1\}) \simeq \mathbb{Z}^2$, and this is not isomorphic to $H_1(\mathbb{C} \setminus \{0, 1, 2\}) \simeq \mathbb{Z}^3$, so $\mathbb{C} \setminus \{0, 1\}$ is not homotopy equivalent to $\mathbb{C} \setminus \{0, 1, 2\}$.

8. (i) False. Define $s_1 : \Delta_1 \to S^1$ by $s_1(t) = e(t)$, so that $u_1 = ([s_1] \bmod B_1 S^1)$ is the usual generator of $H_1 S^1$. Then $c \circ s_1 : \Delta_1 \to S^1$ is a constant path, so $c_*[s_1] \sim 0$, so $c_*(u_1) = 0$ in $H_1 S^1$. Thus c_* is not the identity map on $H_1 S^1$, so c is not homotopic to the identity map on S^1.

(ii) True. Define $s_n = f_n \circ s_1 \colon \Delta_1 \to S^1$, so $s_n(t) = e(t)^n = e(nt)$, so s_n can be unwound to the path $\tilde{s}_n(t) = nt$ in $\mathbb{R}$. It follows that the usual isomorphism $\overline{\phi}\colon H_1S^1 \to \mathbb{Z}$ satisfies

$$\overline{\phi}([s_n] \bmod B_1S^1) = \tilde{s}_n(1) - \tilde{s}_n(0) = n = \overline{\phi}(nu_1),$$

so $f_{n*}(u_1) = nu_1$. It follows that $f_{n*} \neq f_{m*}$ when $n \neq m$, so f_n is not homotopic to f_m when $n \neq m$.

(iii) False. If $f\colon \mathbb{R}^2 \to \mathbb{R}^3$ were a homeomorphism, then it would give a homeomorphism $\mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^3 \setminus \{f(0)\}$. However, $\mathbb{R}^2 \setminus \{0\}$ is homotopy equivalent to S^1 and $\mathbb{R}^3 \setminus \{f(0)\}$ is homeomorphic to $\mathbb{R}^3 \setminus \{0\}$ and thus homotopy equivalent to S^2. We know that $H_1S^1 \simeq \mathbb{Z}$ and $H_1S^2 \simeq 0$ so S^1 is not homotopy equivalent to S^2. It follows that $\mathbb{R}^2 \setminus \{0\}$ is not homotopy equivalent (and thus certainly not homeomorphic) to $\mathbb{R}^3 \setminus \{f(0)\}$, so no such map f can exist.

(iv) False. The map $f_{-1}\colon S^1 \to S^1$ is a homeomorphism and thus a homotopy equivalence, and $(f_{-1})_*(u_1) = -u_1$ so $(f_{-1})_*$ is not the identity map.

9. (i) We say that f and g are homotopic if there exists a continuous map $h\colon I \times X \to Y$ such that $h(0, x) = f(x)$ and $h(1, x) = g(x)$ for all $x \in X$.

(ii) We say that spaces X and Y are homotopy equivalent if there are maps $f\colon X \to Y$ and $g\colon Y \to X$ such that gf is homotopic to 1_X and fg is homotopic to 1_Y.

(iv) For any map $u\colon X \to Y$ we have an induced function $u_*\colon \pi_0X \to \pi_0Y$, given by $u_*\langle x\rangle = \langle u(x)\rangle$ for all $x \in X$. These maps satisfy $1_* = 1$ and $(vu)_* = v_*u_*$, and $u'_* = u_*$ if u' is homotopic to u. If f, g are as above we then have maps $f_*\colon \pi_0X \to \pi_0Y$ and $g_*\colon \pi_0Y \to \pi_0X$, satisfying

$$f_*g_* = (fg)_* = (1_Y)_* = 1_{\pi_0Y}$$

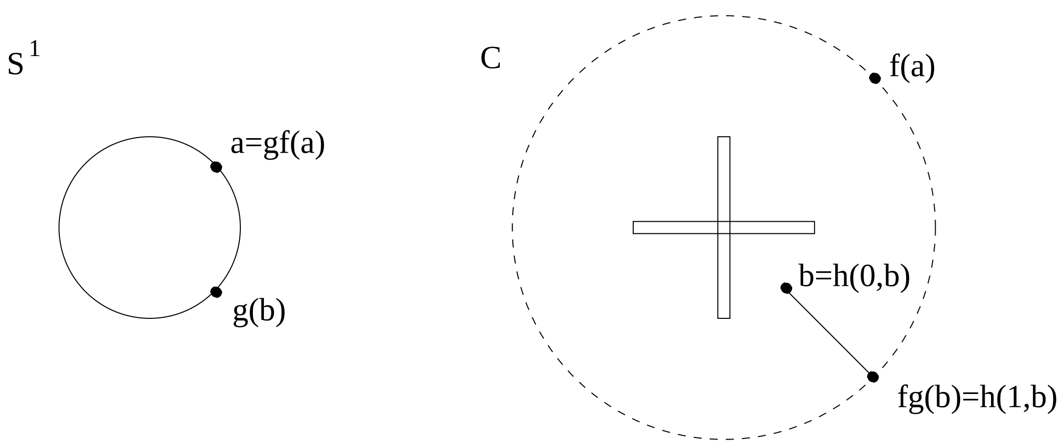
$$g_*f_* = (gf)_* = (1_X)_* = 1_{\pi_0X}.$$

Thus g_* is an inverse for f_*, so f_* is a bijection.

(v) Define maps as follows:

$$\begin{aligned} &f\colon C \to S^1 && f(x,y) = (x,y)/\sqrt{x^2+y^2} \\ &g\colon S^1 \to C && g(x,y) = (2x, 2y) \\ &h\colon I \times C \to C && h(t,x,y) = (1-t)(x,y) + t(x,y)/\sqrt{x^2+y^2}. \end{aligned}$$

Then $fg = 1_{S^1}$, and h is a (linear) homotopy from 1_C to gf, so f is a homotopy equivalence.



10. (i) We say that $U \subseteq X$ is open if for each point $x \in U$, there exists $\epsilon > 0$ such that the open ball $\mathring{B}_\epsilon(x) = \{y \in X \mid d(x,y) < \epsilon\}$ is contained in U.

(ii) We say that $F \subseteq X$ is closed if the complement $X \setminus F$ is open.

(iii) Suppose that F is closed, and that (x_n) is a sequence in F converging to a point $x \in X$. I claim that $x \in F$. If not, then x lies in the open set $X \setminus F$, so there exists $\epsilon > 0$

such that $\overset{\circ}{B}_\epsilon(x) \subseteq X \setminus F$, or equivalently $\overset{\circ}{B}_\epsilon(x) \cap F = \emptyset$. Because $x_n \to x$, there exists N such that $d(x_n, x) < \epsilon$ when $n \geq N$, or in other words $x_n \in \overset{\circ}{B}_\epsilon(x)$ when $n \geq N$. On the other hand, we have $x_n \in F$ for all n by assumption, so for $n \geq N$ we have $x_n \in \overset{\circ}{B}_\epsilon(x) \cap F = \emptyset$, which is impossible. Thus $x \in F$ after all.

Conversely, suppose that F satisfies the condition on sequences; we need to prove that F is closed, or equivalently that $X \setminus F$ is open. If not, then there exists $x \in X \setminus F$ such that $\overset{\circ}{B}_\epsilon(x)$ is not contained in $X \setminus F$ for any $\epsilon > 0$. In particular, $\overset{\circ}{B}_{1/n}(x)$ is not contained in $X \setminus F$, so we can choose a point $x_n \in \overset{\circ}{B}_{1/n}(x) \cap F$. As $x_n \in \overset{\circ}{B}_{1/n}(x)$ we have $d(x_n, x) < 1/n$ so $x_n \to x$. Thus (x_n) is a sequence in F converging to the point x outside F, contradicting the condition on sequences.

11. (i) False. We have $\pi_1 T \simeq \mathbb{Z}^2$, but $\pi_1 S^2 = 0$, so T is not homotopy equivalent to S^2.

(ii) False. Let u be the usual generator of $\pi_1 S^1$. If there were such a map r, we would have $r_* j_* = (rj)_* = 1_* = 1 \colon \pi_1 S^1 \to \pi_1 S^1$, so $u = r_*(j_*(u))$. This is impossible, because B^2 is contractible so $\pi_1 B^2 = \{e\}$ and $j_*(u) \in \pi_1 B^2$ so $j_*(u) = e$ so $r_*(j_*(e)) = e$.

(iii) False. If $f \colon \mathbb{R}^2 \to \mathbb{R}^3$ were a homeomorphism, then it would give a homeomorphism $\mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^3 \setminus \{f(0)\}$. However, $\mathbb{R}^2 \setminus \{0\}$ is homotopy equivalent to S^1 and $\mathbb{R}^3 \setminus \{f(0)\}$ is homeomorphic to $\mathbb{R}^3 \setminus \{0\}$ and thus homotopy equivalent to S^2. We know that $\pi_1 S^1 \simeq \mathbb{Z}$ and $\pi_1 S^2 \simeq 0$ so S^1 is not homotopy equivalent to S^2. It follows that $\mathbb{R}^2 \setminus \{0\}$ is not homotopy equivalent (and thus certainly not homeomorphic) to $\mathbb{R}^3 \setminus \{f(0)\}$, so no such map f can exist.

(iv) True. We can just define $h(t, x) = tf(x)$; this is a homotopy from the constant map with value 0 to f.

12. (i) We say that X is homotopy equivalent to Y if there exist maps $f \colon X \to Y$ and $g \colon Y \to X$ such that $fg \simeq 1_Y$ and $gf \simeq 1_X$ (where $p \simeq q$ means that p is homotopic to q).

Clearly any space X is homotopy equivalent to itself, because we can take $f = g = 1_X$. If X is homotopy equivalent to Y then by reversing the rôles of f and g we see that Y is homotopy equivalent to X.

Now suppose that X is homotopy equivalent to Y and that Y is homotopy equivalent to Z. We can then choose maps f and g as above, and also maps $u \colon Y \to Z$ and $v \colon Z \to Y$ such that $uv \simeq 1_Z$ and $vu \simeq 1_Y$. These give maps $uf \colon X \to Z$ and $gv \colon Z \to X$ such that

$$(uf)(gv) = u(fg)v \simeq u 1_Y v = uv \simeq 1_Z$$

$$(gv)(uf) = g(vu)f \simeq g 1_Y f = gf \simeq 1_X,$$

so X is homotopy equivalent to Z.

This shows that the relation of homotopy equivalence is an equivalence relation.

(ii) A space X is contractible if it is equivalent to the one-point space $\{0\}$. It is path connected if for any two points $x, y \in X$ there is a path $s \colon I \to X$ with $s(0) = x$ and $s(1) = y$.

Suppose that X is contractible, so we have maps $f \colon X \to \{0\}$ and $g \colon \{0\} \to X$ and a homotopy $h \colon 1_X \simeq gf$. Write $a = g(0) \in X$. Note that we must have $f(x) = 0$ for all $x \in X$, because there are no other points in $\{0\}$ that $f(x)$ could be. Thus $gf(x) = a$ for all x. As h is a homotopy from 1 to gf, we have $h(0, x) = x$ and $h(1, x) = a$ for all x. Thus, for any point $x \in X$ we can define a path $s_x \colon I \to X$ by $s_x(t) = h(t, x)$. This starts at x and ands at a. If y is any other point in X we can take the join of s_x with the reverse of s_y to get a path from x to y. Thus X is path connected.

On the other hand, a path connected space need not be contractible. For example, the space S^1 is path-connected (we can define a path from $e^{i\theta}$ to $e^{i\phi}$ by $s(t) = e^{i((1-t)\theta + t\phi)}$) but not contractible (because $\pi_1 S^1 \neq \{e\}$).

(iii) Let $i \colon Y \to X$ be the inclusion, and define $r \colon X \to Y$ by $r(x, y) = (x, \max(0, y))$. We then have $rj = 1$. I claim that if $(x, y) \in X$ then the line segment joining (x, y) to

$jr(x, y)$ is contained in X. If $y \geq 0$ then $rj(x, y) = (x, y)$ and the claim is clear. If $y < 0$ then (as $(x, y) \in X$) we must have $x \in \mathbb{Q}$. We also have $rj(x, y) = (x, 0)$ so the line segment in question is the set of points (x, w) with $y \leq w \leq 0$. As $x \in \mathbb{Q}$, all these points lie in X as required. Thus rj is linearly homotopic to 1_X, which implies that j is a homotopy equivalence.

The set Y is convex and thus contractible, in other words homotopy equivalent to a point. As homotopy equivalence is an equivalence relation, we deduce that X is also homotopy equivalent to a point.