Algebraic Topology Problem Set 1

Please hand in questions 1 and 3 on Tuesday 25th February.

Q1:

Which of the following rules gives a well-defined, continuous function $f\colon X \to Y$? Justify your answer with either a geometric or an algebraic argument.

(a) X is the unit circle centred at the origin in $\mathbb{R}^2$, Y is the circle of radius 3 centred at $(2, 0)$, and $f(x, y)$ is the point where the ray from $(0, 0)$ outwards through (x, y) meets Y.

(b) X is the unit circle centred at the origin in $\mathbb{R}^2$, Y is the line $x = 2$, and $f(x, y)$ is the point where the line through $(0, 0)$ and (x, y) meets Y.

(c) X is the northern hemisphere, Y is the equator, and $f(a)$ is the point where the shortest route along the surface of the earth from a to the south pole crosses Y.

(d) X is the unit circle centred at $(0, 2)$, Y is the x-axis, and $f(a)$ is the point where the vertical line through a crosses Y.

Q2:

(a) Let X be the union of the three axes in $\mathbb{R}^3$, and put $Y = \{z \in \mathbb{C} \mid z^3 \in \mathbb{R}\}$. Explain using pictures why X is homeomorphic to Y.

(b) Let X be the union of the twelve edges of a cube. Draw a subset of $\mathbb{R}^2$ that is homeomorphic to X.

(c) Take a long straight pipe, with walls of zero thickness. Cut a small hole in the side of the pipe and call the resulting surface X. Draw a subset of $\mathbb{R}^2$ that is homeomorphic to X.

(d) Put $X = \{z \in \mathbb{C} \mid 0 < |z| < 1\}$ and $Y = \{z \in \mathbb{C} \mid 1 < |z| < 2\}$. Give a formula for a homeomorphism $f\colon X \to Y$.

Q3:

Which of the following sets in the plane are (i) open, (ii) closed or (iii) neither open nor closed.

(a) the set $A = [0, 1] \times [0, 1)$.

(b) the real axis.

(c) the upper half-plane $B = \{(x, y) \mid y > 0\}$.

(d) the set of points $C = \{(x, y) \mid x \in \mathbb{Q}\}$.

(e) the set of points $D = \{\theta e^{i\theta} \mid \theta \geq 0\}$.

Q4:

Define maps $\mathbb{R}^4 \xrightarrow{f} M_2\mathbb{C} \xrightarrow{g} \mathbb{R}^4$ by

$$f(w, x, y, z) = \begin{pmatrix} w + ix & y + iz \\ -y + iz & w - ix \end{pmatrix}$$

$$g \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (\mathrm{Re}(a), \mathrm{Im}(a), \mathrm{Re}(b), \mathrm{Im}(b)).$$

(a) Show that $g \circ f = 1_{\mathbb{R}^4}$, and deduce that f is injective. Can we also deduce that f is a bijection, with inverse g?

(b) Show that if $(w, x, y, z) \in S^3$ then $f(w, x, y, z) \in SU(2)$, so we can regard f as a map $S^3 \to SU(2)$.

(c) Suppose that $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(2)$. Show that $|a|^2 + |b|^2 = |c|^2 + |d|^2 = ad - bc = 1$ and $a\overline{c} + b\overline{d} = 0$. Deduce that $g(A) \in S^3$, so we can regard g as a map $SU(2) \to S^3$.

(d) Give a formula for $f(g(A))$ (expressed in terms of a and $\overline{a}$ and so on, not in terms of real and imaginary parts).

(e) Calculate $f(g(A)).A^\dagger$.

(f) Deduce that f gives a homeomorphism $S^3 \to SU(2)$.