Algebraic Topology Problem Set 2

Please hand in questions 1,2 and 5 on Tuesday 4th March.

Q1:

Which of the following subsets of $\mathbb{C}$ are compact?

(a) $\{z \in \mathbb{C} \mid f(z) = 0\}$ (where f is a fixed polynomial of degree $d > 0$).
(b) $\{z \in \mathbb{C} \mid \sin(z) = 0\}$.
(b) $\{x + iy \in \mathbb{C} \mid 0 \leq x \leq 1\}$.
(d) $\{z \in \mathbb{C} \mid |z| \leq 1\}$.
(e) $\{z \in \mathbb{C} \mid |z^{10} + z| \leq 1\}$. [Hint: give a lower bound for $|z^{10} + z|$ when $|z| \geq 2$ say.]
(f) $\{z \in \mathbb{C} \mid z^n = 1 \text{ for some } n > 0\}$.

Q2:

Let Y be a metric space, and let $f\colon \mathbb{R}^n \longrightarrow Y$ be continuous. Prove (using facts about compactness) that $f(B^n)$ is a closed subset of Y.

Q3:

Let X be a compact subset of $\mathbb{R}^n$, and let x be a point of X.

(a) Give an example of this situation where $X \setminus \{x\}$ is not compact.
(b) Give an example of this situation where $X \setminus \{x\}$ is compact.
(c) Find a general rule which predicts when $X \setminus \{x\}$ is compact.

Q4:

Let $O(n)$ be the set of $n \times n$ matrices over $\mathbb{R}$ such that $A^T A = I$.

(a) Define $f\colon M_n\mathbb{R} \longrightarrow M_n\mathbb{R}$ by $f(A) = A^T A$. Explain why this is continuous.
(b) If $A \in O(n)$, what can you say about $d_2(A, 0)$?
(c) Prove that $O(n)$ is compact.

Q5:

(a) Give an example of a space X, and a continuous map $f\colon X \longrightarrow \mathbb{R}$ such that $f(x) > 0$ for all $x \in X$ but there is no $\epsilon > 0$ such that $f(x) \geq \epsilon$ for all $x \in X$.
(b) Suppose that X is compact and that $f\colon X \longrightarrow \mathbb{R}$ is continuous and $f(x) > 0$ for all x. Show that there is an $\epsilon > 0$ such that $f(x) \geq \epsilon$ for all x.