Algebraic Topology Problem Set 3

Please hand in questions 2 and 3 on Tuesday 11th March.

Q1:

In this problem we give an alternative way of understanding the map $q\colon S^n \longrightarrow \mathbb{R}P^n$.

Suppose that $A \in M_{n+1}\mathbb{R}$ and $A^2 = A$. Put $U = \{x \in \mathbb{R}^{n+1} \mid Ax = x\}$ and $V = \{x \in \mathbb{R}^{n+1} \mid Ax = 0\}$.

(a) Prove that U and V are vector subspaces of $\mathbb{R}^{n+1}$.
(b) Prove that $\mathbb{R}^{n+1} = U \oplus V$ (or in other words $\mathbb{R}^{n+1} = U + V$ and $U \cap V = \{0\}$). [Hint: $x = Ax + (x - Ax)$].
(c) Now suppose that $A^T = A$. Prove that every vector in U is orthogonal to every vector in V. [Hint: $\langle Au, v\rangle = \langle u, A^T v\rangle$].
(d) Put $k = \dim(U)$, let $\{u_1, \dots, u_k\}$ be a basis for U, and let $\{u_{k+1}, \dots, u_{n+1}\}$ be a basis for V, so that $\{u_1, \dots, u_{n+1}\}$ is a basis for $\mathbb{R}^{n+1}$. We can choose these bases to be normalised, so each u_i is a unit vector. What is Au_i? If P is the matrix whose columns are the vectors u_i, then what is $P^{-1}AP$?
(e) What is $\mathrm{trace}(A)$? [Hint: use (d) and the fact that $\mathrm{trace}(BC) = \mathrm{trace}(CB)$]
(f) Now suppose that $\mathrm{trace}(A) = 1$. Prove that $Au_i = q(u_1)u_i$ for all i and deduce that $A = q(u_1)$.

Q2: Define a continuous map $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ by $f(x) = \exp(-x^2)$.

(a) Sketch the graph of f.
(b) Find an open set $U \subseteq \mathbb{R}$ such that $f(U)$ is not open.
(c) Find a closed set $F \subseteq \mathbb{R}$ such that $f(F)$ is not closed.
(d) Find a compact set $K \subseteq \mathbb{R}$ such that $f^{-1}(K)$ is not compact.
(e) Find sets $A, B \subseteq \mathbb{R}$ such that $f(A \cap B) \neq f(A) \cap f(B)$.

Q3:

(a) Let X be a metric space, and let U and V be open subsets of X such that $X = U \cup V$. Suppose that $f\colon U \longrightarrow Y$ and $g\colon V \longrightarrow Y$ are continuous maps such that $f(x) = g(x)$ for all $x \in U \cap V$. There is then a unique function $h\colon X \longrightarrow Y$ such that $h(x) = f(x)$ for $x \in U$ and $h(x) = g(x)$ for $x \in V$. Prove that h is continuous.
(b) Let X be a metric space, let $U \subseteq X$ be open and let $F \subseteq X$ be closed, and suppose that $X = U \cup F$. Suppose that $f\colon U \longrightarrow Y$ and $g\colon F \longrightarrow Y$ are continuous maps such that $f(x) = g(x)$ for all $x \in U \cap F$, so there is a unique function $h\colon X \longrightarrow Y$ such that $h(x) = f(x)$ for $x \in U$ and $h(x) = g(x)$ for $x \in F$. Give an example to show that h need not be continuous.

Q4: Put $X = \{A \in M_2\mathbb{R} \mid A^2 = A \text{ and } \mathrm{trace}(A) = 1\}$ (so $\mathbb{R}P^1 = \{A \in X \mid A = A^T\}$). In this problem we prove that $X \simeq S^1 \times \mathbb{R}$.

Define $f\colon \mathbb{R}^3 \longrightarrow M_2\mathbb{R}$ by

$$f(x,y,z) = \frac{1}{2}\begin{pmatrix} 1 + x - yz & y + z + xz \\ y - z + xz & 1 - x + yz \end{pmatrix}.$$

(a) Calculate $\mathrm{trace}(f(x,y,z))$ and $\det(f(x,y,z))$ when $(x,y,z) \in S^1 \times \mathbb{R}$.
(b) The Cayley-Hamilton theorem for 2×2 matrices (which you can prove directly for yourself if you wish) says that $A^2 - \mathrm{trace}(A)A + \det(A)I = 0$. Using this, show that f gives a map $S^1 \times \mathbb{R} \longrightarrow X$.
(c) Show that

$$f(\cos(2\theta), \sin(2\theta), z) = R_\theta P(z) R_\theta^{-1}$$

where $P(z) = \begin{pmatrix} 1 & z \\ 0 & 0 \end{pmatrix}$ and $R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$.
(d) Show that if $B \in X$ and $Be_1 = e_1$ then $B = P(z)$ for some z. (Here e_1 is the standard basis vector $(1, 0)$).

(e) Suppose that $A \in X$. Show that there is a unit vector u such that $Au = u$. Show that if $u = (\cos(\theta), \sin(\theta))$ then $R_\theta^{-1} A R_\theta = P(z)$ for some z. Deduce that $f\colon S^1 \times \mathbb{R} \to X$ is surjective.

(f) Define $g\colon M_2\mathbb{R} \to \mathbb{R}^3$ by

$$g\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left(\frac{b^2 - c^2 + 2a - 1}{1 + (b-c)^2}, \frac{2(b - a(b-c))}{1 + (b-c)^2}, b - c \right).$$

Check that $g(f(x, y, z)) = (x, y, z)$. Deduce that $f\colon S^1 \times \mathbb{R} \to X$ is injective and thus bijective. Deduce in turn that $f\colon S^1 \times \mathbb{R} \to X$ is a homeomorphism.

Q5:

Put $SL_2\mathbb{R} = \{A \in M_2\mathbb{R} \mid \det(A) = 1\}$. This evidently contains the group $SO(2)$, which we have seen is homeomorphic to a circle. Here we will show that $SL_2\mathbb{R}$ is homeomorphic to $S^1 \times \mathbb{R}^+ \times \mathbb{R}$ (where $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\} = (0, \infty)$).

Given $(x, y) \in S^1$ and $v \in \mathbb{R}^+$ and $w \in \mathbb{R}$ we put

$$\begin{aligned} R(x,y) &= \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \in SO(2) \subset SL_2\mathbb{R} \\ D(a) &= \begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix} \\ T(b) &= \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \\ f(x, y, a, b) &= R(x, y) D(a) T(b). \end{aligned}$$

(a) Check that all the above matrices lie in $SL_2\mathbb{R}$, so we have a continuous map $f\colon S^1 \times \mathbb{R}^+ \times \mathbb{R} \to SL_2\mathbb{R}$.

(b) Suppose that $B \in SL_2\mathbb{R}$ and $Be_1 = e_1$ (where $e_1 = (1, 0)$ is the first standard basis vector). Prove that $B = T(b)$ for some $b \in \mathbb{R}$.

(c) Suppose that $A \in SL_2\mathbb{R}$. Put $u = Ae_1$ and $a = \|u\|$ and $v = u/a$, so $v \in S^1$. Prove that $D(a)^{-1} R(v)^{-1} A = T(b)$ for some $b \in \mathbb{R}$.

(d) Deduce that f is surjective.

(e) Define $g\colon SL_2\mathbb{R} \to S^1 \times \mathbb{R}^+$ by $g(A) = (Ae_1/\|Ae_1\|, \|Ae_1\|)$. Show that $g(f(u, a, b)) = (u, a)$.

(f) Deduce that if $f(u, a, b) = f(u', a', b')$ then $u = u'$ and $a = a'$; then deduce in turn that $b = b'$ as well. This shows that f is injective and thus bijective.

(g) Suppose that $A_n \to A$ in $SL_2\mathbb{R}$, and put $f^{-1}(A_n) = (u_n, a_n, b_n)$ and $f^{-1}(A) = (u, a, b)$. Using g show that $u_n \to u$ and $a_n \to a$. By considering the matrices $B_n = D(a_n)^{-1} R(u_n)^{-1} A_n$, deduce that $b_n \to b$. Now deduce that f is a homeomorphism.