Algebraic Topology Problem Set 4

No questions to be handed in this week.

Q1: Put $X = \{(x, y) \in \mathbb{R}^2 \mid y^2 = x^3 - x\}$. (This is an example of an *elliptic curve*; such curves are important in number theory and certain other areas of mathematics.)

Sketch the graph of the function $x^3 - x$, then draw a picture of X. By considering the function $f : X \longrightarrow \mathbb{R}$ given by $f(x, y) = 2x - 1$, show that X is not path-connected.

Q2: For each of the following pairs of spaces, either give a continuous surjective map $f : X \longrightarrow Y$, or prove that no such map exists.

(a) $X = \mathbb{R}$ and $Y = S^1$
(b) $X = \mathbb{C}$ and $Y = \mathbb{C} \setminus \{0\}$
(c) $X = \mathbb{R}$ and $Y = \mathbb{R} \setminus \{0\}$
(d) $X = S^2$ and $Y = S^2 \setminus \{\text{north pole}\}$
(e) $X = S^2$ and $Y = S^1$.

Q3: View each of the capital letters A,B,C,D,E,F as closed subsets of the plane. Consider the following numbers associated to a metric space X:

(i) $a(X)$ is the greatest number of points that can be removed without disconnecting X and
(ii) $b(X)$ is the least number of points that need to be removed to disconnect the space X.

(For example $a($ the letter A$) = 3$, since we can remove the two end points at the feet, and one point around the top circuit, but every 4 points disconnect A; it is clear that $b(A) = 1$, since any point (except an endpoint) in either of the legs disconnects the letter.)

(a) Find the values of the invariants on the specified letters.
(b) Group the letters according to the values of the invariants. For every pair of letters with the same values of a and b, either indicate that they are homeomorphic or define further invariants to show that they are not.
(c) Summarise your work in (b) in the form of a classification of the letters up to homeomorphism.

Q4: Consider the following spaces:

$$X_0 = \text{ the union of all the edges of a cube}$$
$$X_1 = \{(x, y) \in [-1, 1] \times [-1, 1] \mid x \in \mathbb{Z} \text{ or } y \in \mathbb{Z}\}$$
$$X_2 = \{(x, y, z) \in S^2 \mid xyz = 0\}$$

Draw pictures of these spaces (or of subsets of $\mathbb{R}^2$ that are homeomorphic to them). Evaluate $a(X_i)$ for $i = 0, 1, 2$, and deduce that none of the spaces are homeomorphic to each other.

Q5: In this problem, we'll show that the space $GL_n(\mathbb{C})$ (of invertible $n \times n$ matrices over the complex numbers) is path-connected.

Let A be an $n \times n$ invertible complex matrix, with eigenvalues $\lambda_1, \ldots, \lambda_r$ say. Because A is invertible, we know that $\lambda_i \neq 0$ for all i. Define

$$L_i = \{-t\lambda_i \mid 0 \leq t < \infty\} \subset \mathbb{C},$$

so L_i is the half-line starting at 0 and passing through $-\lambda_i$.

1. Show that the eigenvalues of $\alpha A + \beta I$ are $\alpha\lambda_1 + \beta, \ldots, \alpha\lambda_r + \beta$.
2. Let μ be a nonzero complex number. Show that the linear path from λ_i to μ passes through 0 iff $\mu \in L_i$ (think geometrically).
3. Show that the linear path from A to μI in $M_n\mathbb{C}$ actually lies in $GL_n(\mathbb{C})$ iff $\mu \notin L_i$ for all i.
4. Deduce that A can be connected in $GL_n(\mathbb{C})$ to some matrix of the form μI with $\mu \neq 0$.
5. Show that all matrices of the form μI can be connected to I in $GL_n(\mathbb{C})$.
6. Deduce that $GL_n(\mathbb{C})$ is connected.