Algebraic Topology Problem Set 5

Please hand in questions 1,2 and 3 on Tuesday 25th March.

Q1: Let $f\colon X \longrightarrow \mathbb{R} \setminus \{0\}$ be a continuous map. Show that f is homotopic to a map g such that $g(x)^2 = 1$ for all $x \in X$.

Q2: Recall that for any space X and any $x \in X$ we have a constant path $c_x\colon I \longrightarrow X$ given by $c_x(t) = x$ for all t.

(a) Let $u\colon I \longrightarrow X$ be a path. Prove that u is homotopic to $c_{u(0)}$.

(b) Suppose that X is path-connected. Prove that any two paths $u, v\colon I \longrightarrow X$ are homotopic to each other.

Q3: Regard S^1 as a subset of the xy-plane in $\mathbb{R}^3$, so $S^1 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2 = 1 \text{ and } z = 0\}$. Put $X = \mathbb{R}^3 \setminus S^1$. Define $u, v, w\colon S^1 \longrightarrow \mathbb{R}^3$ by

$$\begin{aligned} u(x,y) &= (x/2+1, 0, y/2) \\ v(x,y) &= (-x/2-1, 0, y/2) \\ w(x,y) &= (x/2-1, 0, -y/2) \end{aligned}$$

(a) Draw a picture of u and v.

(b) Show that u and v, regarded as maps $S^1 \longrightarrow \mathbb{R}^3$, are homotopic to each other. (This is easy.)

(c) Check that $u(x,y)$, $v(x,y)$ and $w(x,y)$ lie in X, so we can regard u, v and w as maps $S^1 \longrightarrow X$.

(d) Show that u and v are still homotopic to each other when regarded as maps $S^1 \longrightarrow X$.

(e) Give a homotopy $m\colon I \times S^1 \longrightarrow X$ between v and w with the property that $m(I \times S^1) = v(S^1)$.

Q4: Let f be a polynomial of degree n over $\mathbb{R}$, and suppose that f has n distinct real roots. Put $X = \{x \in \mathbb{R} \mid f(x) \neq 0\}$ and $Y = \{x \in \mathbb{R} \mid f'(x) = 0\}$. How far is X from being homotopy equivalent to Y? What happens if some of the roots of f are not real, or if there are repeated roots?

Q5: Show that the space $X = S^2 \setminus S^1$ of points of the 2-sphere not on the equator is homotopy equivalent to the space $Y = \{N, S\}$ consisting of the north and south poles.

Q6: Define $f\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ by $f(x,y) = \cos(\sqrt{x^2+y^2})$. Let X be the set of points where $f \neq 0$, and let Y be the set of points where $\partial f/\partial r = 0$. (The value of $\partial f/\partial r$ at $(0,0)$ is not really well-defined, but we will take it to be 0.) Prove that X is homotopy equivalent to Y.

Q7:

In classifying letters and other one dimensional objects, we have made much use of the number of endpoints, and the number of points where three lines meet (ie Y junctions), and so forth.

(i) Give a definition of 'endpoint' which is topological (in the sense that if $f\colon X \longrightarrow Y$ is a homeomorphism and $x \in X$ is an endpoint, then $f(x)$ is also an endpoint).

(ii) Let us say a point is a 'multiple point' if three or more lines meet there. Give a topological definition of 'multiple point'.

[Hint: the property of a point x we are seeking to isolate concerns what the space looks like in small open sets containing x.]