Algebraic Topology Problem Set 6

Please hand in questions 1,2 and 3 on Tuesday April 1st.

Q1: Let X and Y be spaces such that $\pi_0 X$ and $\pi_0 Y$ are finite sets. Suppose that $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow X$ are continuous maps such that gf is homotopic to 1_X.

(a) Show that $f_*\colon \pi_0 X \longrightarrow \pi_0 Y$ is injective.
(b) Show that $g_*\colon \pi_0 Y \longrightarrow \pi_0 X$ is surjective.
(c) Let A and B be finite sets. Show that if there is an injective map $j\colon A \longrightarrow B$, or a surjective map $q\colon B \longrightarrow A$, then $|A| \leq |B|$. Deduce that $|\pi_0 X| \leq |\pi_0 Y|$.

Of course we deduce from this that $|\pi_0 X| \leq |\pi_0 Y|$.

Q2:

(a) Suppose that X is contractible, so there exists a point $a \in X$ and a map $h\colon I \times X \longrightarrow X$ such that $h(0,x) = x$ and $h(1,x) = a$ for all $x \in X$. In lectures we gave an indirect proof that X is path-connected. Given points $x, y \in X$, find formulae (in terms of h) for (i) a path from x to a; (ii) a path from y to a; (iii) a path from x to y.
(b) Now suppose we have maps $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow X$ and a homotopy $k\colon I \times Y \longrightarrow Y$ between 1_Y and fg. Find a point $b \in Y$ and a contraction of Y to b.

Q3: Put $M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Consider another matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2\mathbb{C}$.

(a) For which matrices A do we have $AM = MA$? Assuming that $AM = MA$, factorise $\det(A)$.
(b) Put $X = \{A \in GL_2\mathbb{C} \mid AM = MA\}$. Show that the map $f\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a+b, a-b)$ gives a homeomorphism $X \longrightarrow (\mathbb{C} \setminus \{0\}) \times (\mathbb{C} \setminus \{0\})$.
(c) Deduce that X is homotopy equivalent to the torus.

Q4: In this problem we study the topology of some spaces occurring in the theory of relativity. Any event in the history of the universe can be located by giving the time t and position (x, y, z) at which it occurs. (Of course, we need to make some agreement about where the origin and the axes are before we can do this.)

Every space described in this problem is either contractible or homotopy equivalent to S^d for some d. When asked to "classify" a space, you should either say that it is contractible, or say what the appropriate d is. Remember that S^0 is the space with two points.

We put

$$\begin{aligned}
L &= \text{ the light cone} \\
&= \{(t,x,y,z) \in \mathbb{R}^4 \mid t^2 - x^2 - y^2 - z^2 = 0\} \\
L^+ &= \text{ the forward light cone} \\
&= \{(t,x,y,z) \in L \mid t > 0\} \\
C &= \text{ the celestial sphere} \\
&= \{(t,x,y,z) \in L^+ \mid t = 1\} \\
H &= \{(t,x,y,z) \in \mathbb{R}^4 \mid t^2 - x^2 - y^2 - z^2 = 1\} \\
H^+ &= \{(t,x,y,z) \in H \mid t > 0\} \\
T &= \{(t,x,y,z) \in \mathbb{R}^4 \mid t^2 - x^2 - y^2 - z^2 = -1\}.
\end{aligned}$$

We refer to all this as the case $n = 3$, because we are using three space coordinates: x, y and z. To help us see what is going on, we will also study the cases $n = 2$ (where we ignore z) and $n = 1$ (where we ignore both y and z). Thus, for example, in the case $n = 1$ we have $L = \{(t,x) \in \mathbb{R}^2 \mid t^2 - x^2 = 0\}$, and in the case $n = 2$ we have $H = \{(t,x,y) \in \mathbb{R}^3 \mid t^2 - x^2 - y^2 = 1\}$.

(a) In the case $n = 1$, draw the subsets L, L^+, C, H, H^+ and T of $\mathbb{R}^2$, and classify them. You should do this "by inspection"; no proof is required.
(b) Now do the same in the case $n = 2$.

(c) In the case $n = 3$, classify the spaces L, L^+, C, H, H^+ and T, giving formulae and proofs.