Algebraic Topology Problem Set 7

Please hand in questions 2, 3 and 4 on Tuesday 6th May.

Q1: Let X be a metric space, and let $f, g\colon X \longrightarrow S^1$ be continuous maps. Suppose that f is linearly homotopic to g. What can you deduce?

Q2: Put

$$X = \{z \in \mathbb{C} \mid 1 < |z| < 5\}$$
$$Y = S^1 = \{z \in \mathbb{C} \mid |z| = 1\}.$$

Define maps $f\colon X \longrightarrow Y$ and $g_0, g_1\colon Y \longrightarrow X$ by

$$f(z) = z/|z|$$
$$g_0(z) = 3z$$
$$g_1(z) = 3 + z$$

Draw pictures.

Are the following statements true, false, or meaningless? Justify your answers.

(a) f is homotopic to g_0
(b) fg_0 is homotopic to 1_X
(c) fg_0 is homotopic to 1_Y
(d) $g_0 f$ is homotopic to 1_X
(e) $g_1 f$ is homotopic to 1_X

Q3: Suppose we have maps $W \xrightarrow{f} X \xrightarrow{g} Y$, and that X is contractible.

(a) Prove that f is homotopic to a constant map.
(b) Prove that g is homotopic to a constant map.

Q4: Find examples of the following things.

(a) A space X, another space $Y \subseteq \mathbb{R}^2$, and two maps $f, g\colon X \longrightarrow Y$ such that f is homotopic to g, but not linearly homotopic to g.
(b) A connected space X and contractible subspaces $Y, Z \subseteq X$ such that $Y \cap Z$ is connected but not contractible.
(c) Spaces X, Y and an injective map $f\colon X \longrightarrow Y$ such that $f_*\colon \pi_0 X \longrightarrow \pi_0 Y$ is not injective.
(d) Spaces X, Y and a surjective map $f\colon X \longrightarrow Y$ such that $f_*\colon \pi_1 X \longrightarrow \pi_1 Y$ is not surjective.

Q5: Let a, b, c, d be points in a space X, and suppose we have paths u, v, w in X with u running from a to b, v from b to c, and w from c to d.

Define $z\colon [0, 3] \longrightarrow X$ by

$$z(t) = \begin{cases} 0 \le t \le 1 & u(t) \\ 1 \le t \le 2 & v(t-1) \\ 2 \le t \le 3 & w(t-2). \end{cases}$$

Find functions $f, g\colon I \longrightarrow [0, 3]$ such that $((u * v) * w) = z \circ f$ and $(u * (v * w)) = z \circ g$, and plot their graphs. Using this, give an alternative proof that $u * (v * w) \simeq_{\text{re}} (u * v) * w$.