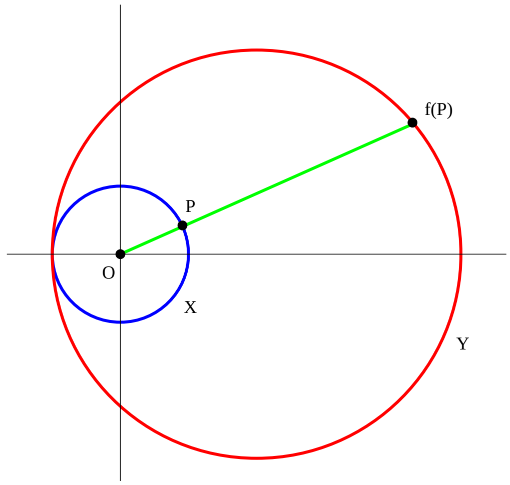
Algebraic Topology Problem Set 1 — Solutions

Q1:

(a) The picture is:

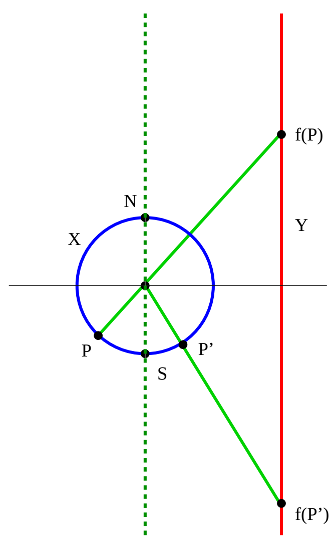


It is geometrically clear that as the point P moves continuously, so does the point $f(P)$. More formally, we have $f(x,y) = t(x,y) = (tx,ty)$ for some $t > 0$. As $f(x,y)$ lies on the circle of radius 3 centred at $(2,0)$, we have $(tx-2)^2 + (ty)^2 = 9$, or in other words $(x^2+y^2)t^2 - 4xt + 4 = 9$. As $x^2+y^2 = 1$ this simplifies to $t^2 - 4xt - 5 = 0$, which can be solved to give $t = 2x \pm \sqrt{4x^2+5}$. Moreover, we note that $4x^2+5 > (2x)^2$ so $\sqrt{4x^2+5} > 2x$ so $2x - \sqrt{4x^2+5} < 0$, but $t > 0$ so $t = 2x + \sqrt{4x^2+5}$. Thus

$$f(x,y) = (2x + \sqrt{4x^2+5})(x,y),$$

which is continuous.

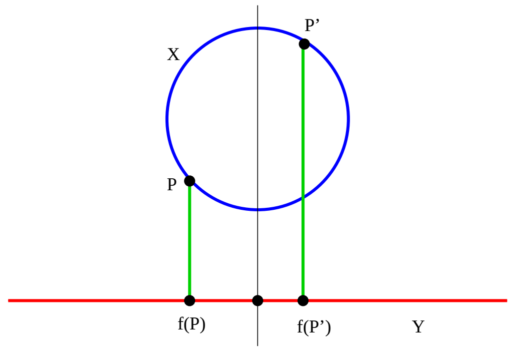
(b) The picture is:



Of course, $f(N)$ and $f(S)$ are undefined, because the dotted line does not meet Y. Thus, f does not give a well-defined, continuous function. For points other than N and S, the function f is well-defined and given by $f(x,y) = (2, 2y/x)$.

(c) Here, if N is the north pole, then all great circle routes from N to S have the same length, and they cross the equator in different places. Thus, $f(N)$ is not well-defined.

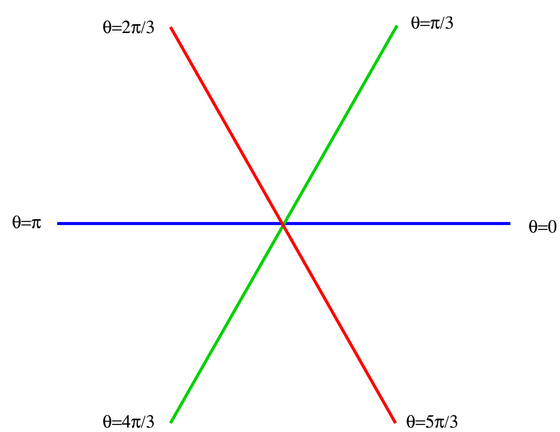
(d) The picture is:



Here $f(x,y)$ is just $(x,0)$, so f is clearly continuous.

Q2:

(a) We can write any complex number $z \neq 0$ in the form $re^{i\theta}$ with $r > 0$ and $0 \leq \theta < 2\pi$. We then have $z^3 = r^3 e^{3i\theta}$, and for this to be real, 3θ must have the form $n\pi$ for some integer n. (Here z^3 is positive real if n is even, and negative real if n is odd.) As $0 \leq \theta < 2\pi$, the possibilities are $\theta = 0, \pi/3, 2\pi/3, \pi, 4\pi/3$ or $5\pi/3$. Of course, the point $z = 0$ also lies in Y. Thus, Y looks like this:

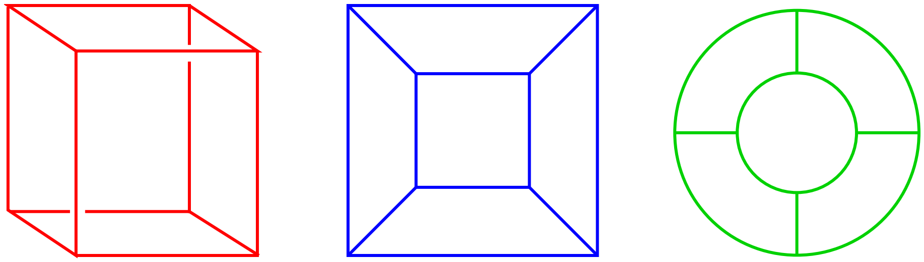


We can define a homeomorphism $f\colon X \longrightarrow Y$ by

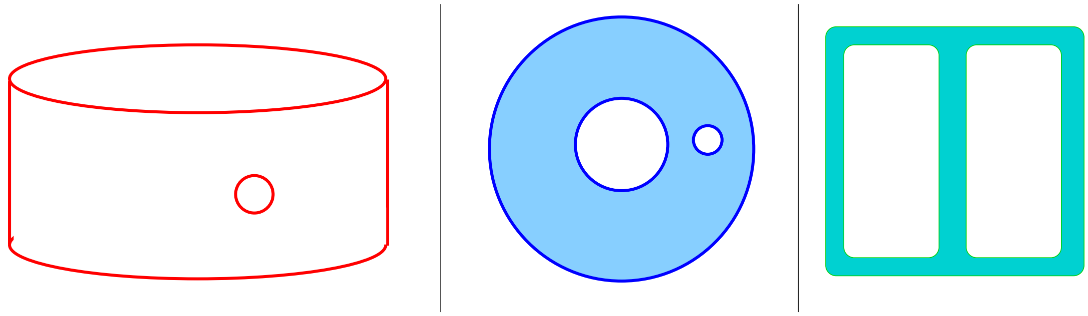
$$f(x,0,0) = x$$
$$f(0,y,0) = ye^{i\pi/3}$$
$$f(0,0,z) = ze^{2i\pi/3}.$$

Thus, f sends the positive x, y and z-axes to the rays where $\theta = 0, \pi/3$ and $2\pi/3$ respectively. It also sends the negative x, y and z-axes to the rays where $\theta = \pi, 4\pi/3$ and $5\pi/3$ respectively.

(b) The picture on the left shows X; in the middle and on the right we have two different subsets of $\mathbb{R}^2$ that are homeomorphic to X.



(c) X is homeomorphic to a very short fat pipe with a hole removed, as shown on the left. This can be opened out and laid flat as shown in the middle. This in turn is homeomorphic to the more symmetrical space shown on the right.



(d) The simplest answer is to take $f(re^{i\theta}) = (r+1)e^{i\theta}$, so $f^{-1}(se^{i\theta}) = (s-1)e^{i\theta}$. If $z = re^{i\theta}$ then $r = |z|$ and $e^{i\theta} = z/|z|$. Using this, we can rewrite f in different notation as follows:

$$f(z) = (|z|+1)z/|z| = z + z/|z|.$$

Q3:

(a) The set A is neither open nor closed. We have $(0,0) \in A$ but no disc $\mathring{B}_\epsilon(0,0)$ is contained in A so A is not open. The sequence $(0, 1-1/n)$ lies in A and converges in $\mathbb{R}^2$ to $(0,1)$, which does not lie in A; so A is not closed.

(b) The real line $L = \{(x,0) \mid x \in \mathbb{R}\}$ is closed but not open in $\mathbb{R}^2$. It is not open because $(0,0) \in L$ but no disc $\mathring{B}_\epsilon(0,0)$ is contained in L. To see that it is closed, suppose we have a convergent sequence $(x_n, y_n) \longrightarrow (x,y)$ in $\mathbb{R}^2$ (so $x_n \longrightarrow x$ and $y_n \longrightarrow y$) and $(x_n, y_n) \in L$ for all n. This means that $y_n = 0$ for all n and $y_n \longrightarrow y$ so $y = 0$ so $(x,y) \in L$.

(c) The set B is open but not closed. To see that it is open, consider a point $(x,y) \in B$, and put $\epsilon = y > 0$. If $(x', y') \in B_\epsilon(x,y)$ then $|y - y'|^2 \leq (x-x')^2 + (y-y')^2$, so

$$|y' - y| \leq \sqrt{(x'-x)^2 + (y'-y)^2} = d_2((x', y'), (x,y)) < \epsilon,$$

so

$$y' \geq y - |y' - y| > y - \epsilon = 0,$$

so $(x', y') \in B$. This means that $B_\epsilon(x,y) \subseteq B$ as required.

To see that B is not closed, note that $(0, 1/n) \in B$ and $(0, 1/n) \longrightarrow (0,0)$ in $\mathbb{R}^2$ but $(0,0) \notin B$.

(d) The set C is neither open nor closed. To see this, let a be your favourite irrational number. Choose a sequence of rational numbers a_n converging to a; for example, we could let a_n be the number obtained by truncating the decimal expansion of a at the n'th decimal place. Then $(a_n, 0) \in C$ and $(a_n, 0) \longrightarrow (a, 0)$ but $(a, 0) \notin C$, so C is not closed. I also claim that no disc $\mathring{B}_\epsilon(0,0)$ is contained in C (and as $(0,0) \in C$ this proves that C is not open). To see this, note that when n is large enough we have $|a/n| < \epsilon$ but a/n is irrational; thus $(a/n, 0) \in B_\epsilon(0,0)$ but $(a/n, 0) \notin C$.

(e) The set D is closed but not open. The easiest way to see this is to define $f(z) = |z|e^{i|z|} - z$, so $f \colon \mathbb{C} \longrightarrow \mathbb{C}$ is continuous. If $z \in D$ it is easy to see that $f(z) = 0$. Conversely, if $f(z) = 0$ then $z = \theta e^{i\theta}$ where $\theta = |z|$. Thus, we see that $D = \{z \mid f(z) = 0\}$. If $z_n \longrightarrow z$ in $\mathbb{C}$ and $z_n \in D$ for all n then $f(z_n) = 0$ and $f(z_n) \longrightarrow f(z)$ (because f is continuous) and thus $f(z) = 0$, so $z \in D$. This shows that D is closed. On the other hand, suppose $\epsilon > 0$. Put $w = -\epsilon e^{i\epsilon/2}/2$, so $|w| < \epsilon$ but $|f(w)| = \epsilon > 0$ so $w \notin D$. This shows that no disc $B_\epsilon(0,0)$ is contained in D, so D is not open.

Q4:

(a) We have

$$\begin{aligned} g(f(w,x,y,z)) &= g\begin{pmatrix} w+ix & y+iz \\ -y+iz & w-ix \end{pmatrix} \\ &= (\operatorname{Re}(w+ix), \operatorname{Im}(w+ix), \operatorname{Re}(y+iz), \operatorname{Im}(y+iz)) \\ &= (w,x,y,z) \end{aligned}$$

so $g \circ f = 1$. Thus, if $p, q \in \mathbb{R}^4$ and $f(p) = f(q)$ then $g(f(p)) = g(f(q))$ or in other words $p = q$, showing that f is injective.

We cannot immediately conclude that $f\colon \mathbb{R}^4 \longrightarrow M_2\mathbb{C}$ is a bijection, because $f \circ g \neq 1_{M_2\mathbb{C}}$. Indeed, you can easily check that $f\left(g\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)\right) = \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$. This is reasonable, because $M_2\mathbb{C}$ is homeomorphic to $\mathbb{C}^4$ and thus to $\mathbb{R}^8$; you will probably believe me when I tell you that there is no homeomorphism $\mathbb{R}^4 \simeq \mathbb{R}^8$, although this is in fact quite hard to prove.

(b) Suppose that $(w, x, y, z) \in S^3$, so $w^2 + x^2 + y^2 + z^2 = 1$. Put $a = w + ix$ and $b = y + iz$ so $\overline{a} = w - ix$ and $\overline{b} = y - iz$ so $-\overline{b} = -y + iz$. Put

$$A = f(w, x, y, z) = \begin{pmatrix} w + ix & y + iz \\ -y + iz & w - ix \end{pmatrix} = \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix}.$$

We have

$$\det(A) = a\overline{a} + b\overline{b} = |a|^2 + |b|^2 = w^2 + x^2 + y^2 + z^2 = 1$$

and

$$A^\dagger = \begin{pmatrix} \overline{a} & -b \\ \overline{b} & a \end{pmatrix}$$

so

$$\begin{aligned} AA^\dagger &= \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix} \begin{pmatrix} \overline{a} & -b \\ \overline{b} & a \end{pmatrix} \\ &= \begin{pmatrix} a\overline{a} + b\overline{b} & -ab + ba \\ -\overline{b}\overline{a} + \overline{a}\overline{b} & \overline{b}b + \overline{a}a \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

Thus $A^\dagger = A^{-1}$, so $A \in SU(2)$. This shows that f gives a map $f\colon S^3 \longrightarrow SU(2)$.

(c) Suppose that $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2\mathbb{C}$. Then $\det(A) = ad - bc$ and $A^\dagger = \begin{pmatrix} \overline{a} & \overline{c} \\ \overline{b} & \overline{d} \end{pmatrix}$ so

$$AA^\dagger = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \overline{a} & \overline{c} \\ \overline{b} & \overline{d} \end{pmatrix} = \begin{pmatrix} a\overline{a} + b\overline{b} & a\overline{c} + b\overline{d} \\ c\overline{a} + d\overline{b} & c\overline{c} + d\overline{d} \end{pmatrix} = \begin{pmatrix} |a|^2 + |b|^2 & a\overline{c} + b\overline{d} \\ \overline{a\overline{c} + b\overline{d}} & |c|^2 + |d|^2 \end{pmatrix}$$

Recall that $A \in SU(2)$ iff $\det(A) = 1$ and $AA^\dagger = I$. This is clearly equivalent to $ad - bc = |a|^2 + |b|^2 = |c|^2 + |d|^2 = 1$ and $a\overline{c} + b\overline{d} = 0$.

If this holds, then

$$\|g(A)\|^2 = \mathrm{Re}(a)^2 + \mathrm{Im}(a)^2 + \mathrm{Re}(b)^2 + \mathrm{Im}(b)^2 = |a|^2 + |b|^2 = 1,$$

so $g(A) \in S^3$ as claimed.

(d) We have

$$\begin{aligned} f\left(g\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) &= f(\mathrm{Re}(a), \mathrm{Im}(a), \mathrm{Re}(b), \mathrm{Im}(b)) \\ &= \begin{pmatrix} \mathrm{Re}(a) + i\mathrm{Im}(a) & \mathrm{Re}(b) + i\mathrm{Im}(b) \\ -\mathrm{Re}(b) + i\mathrm{Im}(b) & \mathrm{Re}(a) - i\mathrm{Im}(a) \end{pmatrix} \\ &= \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix} \end{aligned}$$

(e) It follows that

$$\begin{aligned} f(g(A)).A^\dagger &= \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix} \begin{pmatrix} \overline{a} & \overline{c} \\ \overline{b} & \overline{d} \end{pmatrix} \\ &= \begin{pmatrix} |a|^2 + |b|^2 & a\overline{c} + b\overline{d} \\ -\overline{b}\overline{a} + \overline{a}\overline{b} & \overline{ad - bc} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

(e) As $A^\dagger = A^{-1}$, we can rewrite the above as $f(g(A))A^{-1} = I$, so $f(g(A)) = A$. In conjunction with (a), this means that $f\colon S^3 \longrightarrow SU(2)$ is the inverse of $g\colon SU(2) \longrightarrow S^3$. It is clear that both f and g are continuous, so they are homeomorphisms.