Algebraic Topology Problem Set 2 — Solutions

Q1:

(a) This set is finite and thus compact.
(b) This set is $\{n\pi \mid n \in \mathbb{Z}\}$, which is unbounded and thus not compact. The sequence $\pi, 2\pi, 3\pi, \dots$ has no convergent subsequence.
(c) This set is again unbounded and thus not compact. The sequence $i, 2i, 3i, \dots$ has no convergent subsequence.
(d) This set is bounded and closed and thus compact.
(e) Put $X = \{z \mid |z^{10} + z| \leq 1\}$. When $|z| \geq 2$ we have

$$|z^{10} + z| \geq |z|^{10} - |z| = |z|(|z|^9 - 1) \geq 2.(2^9 - 1) = 1022 > 1.$$

Thus, if $z \in X$ we must have $|z| < 2$, so X is bounded. If (z_n) is a sequence in X converging to some $z \in \mathbb{C}$, then clearly $1 \geq |z_n^{10} + z_n| \longrightarrow |z^{10} + z|$ so $|z^{10} + z| \leq 1$ so $z \in X$. This shows that X is also closed, so it is compact.
(f) Put $Y = \{z \in \mathbb{C} \mid z^n = 1 \text{ for some } n > 0\}$. Suppose that $z \in Y$, so $z^n = 1$ for some $n > 0$. Then $|z|^n = 1$ and $|z|$ is a nonnegative real number so $|z| = 1$. Thus $z = e^{2\pi i t}$ for some $t \in \mathbb{R}$, and $e^{2\pi i n t} = 1$. This means that $m := nt$ must be an integer, and thus $t = m/n$ is rational. Conversely, suppose that s is rational and put $w = e^{2\pi i s}$. We can write $s = p/q$ with $p, q \in \mathbb{Z}$ and $q > 0$, and we find that $w^q = e^{2\pi i p} = 1$, so $w \in Y$. Thus $Y = \{e^{2\pi i s} \mid s \in \mathbb{Q}\}$. Now choose an irrational number s and a sequence of rational numbers $s_1, s_2, \dots$ converging to s. Put $w = e^{2\pi i s}$ and $w_k = e^{2\pi i s_k}$. Then $w_k \in Y$ and $w_k \longrightarrow w$ but $w \notin Y$. Thus Y is not closed, and thus not compact.

Q2:

The set $B^n \subset \mathbb{R}^n$ is bounded and closed, hence compact. We know that continuous images of compact sets are compact, so $Z := f(B^n)$ is a compact subset of Y. This implies that Z must be closed. Indeed, suppose we have a sequence z_k in Z, converging to some point $y \in Y$. Then by compactness, some subsequence z_{k_j} converges to some point $z \in Z$, but the whole sequence converges to y so the subsequence must converge to y as well, so $y = z$, so $y \in Z$. Thus Z is closed as claimed.

Q3:

(a) $[0, 1]$ is compact and $0 \in [0, 1]$ but $[0, 1] \setminus \{0\}$ is not compact.
(b) $\{0, 1\}$ is compact and $0 \in \{0, 1\}$ and $\{1\} = \{0, 1\} \setminus \{0\}$ is also compact.
(c) In general, if $X \subset \mathbb{R}^n$ is compact and $x \in X$ then $X \setminus \{x\}$ is compact iff x is an *isolated point*, in other words there exists $\epsilon > 0$ such that $d(x, y) > \epsilon$ for all $y \in X \setminus \{x\}$.

Q4:

(a) We have

$$f(A)_{ij} = (A^T A)_{ij} = \sum_{k=1}^{n} A^T_{ik} A_{kj} = \sum_{k=1}^{n} A_{ki} A_{kj}.$$

This is a polynomial function of the entries A_{pq}, so f is continuous.
(b) Recall that the trace of a matrix is the sum of the diagonal entries. In particular, if I is the $n \times n$ identity matrix, with n ones on the diagonal, we have $\text{trace}(I) = n$. It follows that

$$d_2(A, 0) = \sqrt{\text{trace}(A^T A)} = \sqrt{\text{trace}(I)} = \sqrt{n}.$$

(c) It is immediate from (b) that $O(n)$ is bounded. Next, suppose we have a sequence (A_k) in $O(n)$ and $A_k \longrightarrow A$ for some $A \in M_n\mathbb{R}$. Then $f(A_k) = I$ for all k, and f is continuous so $f(A) = \lim_{k \longrightarrow \infty} f(A_k) = I$, so $A \in O(n)$. This proves that $O(n)$ is closed in $\mathbb{R}^{n^2}$ as well as being bounded, so it is compact.

Q5:

(a) Just take $X = (0, \infty)$ and $f(x) = x$. Clearly $f(x) > 0$ for all $x \in X$. However, if $\epsilon > 0$ then $\epsilon/2 \in X$ and $f(\epsilon/2) = \epsilon/2 < \epsilon$, so it is not true that $f(x) \geq \epsilon$ for all $x \in X$.

(b) Now suppose that X is compact and $f\colon X \longrightarrow \mathbb{R}$ is continuous and $f(x) > 0$ for all $x \in X$. Consider the function $g(x) = 1/f(x)$, which is defined and continuous everywhere on X because $f(x) > 0$ for all x. We know that a continuous function from a compact space to $\mathbb{R}$ is bounded, so there is some constant $C > 0$ such that $g(x) \leq C$ for all x. Now put $\epsilon = 1/C$ and observe that $f(x) = 1/g(x) \geq 1/C = \epsilon$ for all x.