Algebraic Topology Problem Set 3 — Solutions

Q1:

(a) Suppose that $x, y \in U$ and $a, b \in \mathbb{R}$. Then $Ax = x$ and $Ay = y$ so $A(ax+by) = aAx+bAy = ax+by$, so $ax+by \in U$. Visibly we also have $0 \in U$, so U is a vector subspace.

Now suppose instead that $x, y \in V$ and $a, b \in \mathbb{R}$. Then $Ax = Ay = 0$ so $A(ax+by) = aAx + bAy = 0+0 = 0$ so $ax+by \in V$. It is also clear that $0 \in V$, so V is again a vector subspace.

(b) Given $x \in \mathbb{R}^{n+1}$, put $u = Ax$ and $v = x - Ax$. Then $Au = A^2x = Ax = u$ (because $A^2 = A$) so $u \in U$. Similarly, $Av = Ax - A^2x = Ax - Ax = 0$, so $v \in V$. Clearly $x = u+v$, so $x \in U+V$. This works for any x, so $\mathbb{R}^{n+1} = U+V$. On the other hand, if $x \in U \cap V$ then $x = Ax$ (as $x \in U$) and $Ax = 0$ (as $x \in V$) so $x = 0$. Thus $U \cap V = \{0\}$ as required.

(c) Suppose that $u \in U$ and $v \in V$. Then

$$\langle u, v\rangle = \langle Au, v\rangle = \langle u, A^T v\rangle = \langle u, Av\rangle = \langle u, 0\rangle = 0$$

as claimed. (The first equality uses the fact that $Au = u$, the third uses the fact that $A = A^T$, and the fourth uses $Av = 0$.)

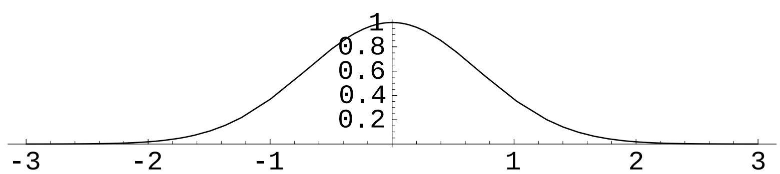
(d) If $i \leq k$ then $u_i \in U$ so $Au_i = u_i$. If $i > k$ then $u_i \in V$ so $Au_i = 0$. It is a standard fact of linear algebra that $P^{-1}AP$ is the matrix that expresses the action of A on the basis $\{u_1, \dots, u_{n+1}\}$, so it consists of a $k \times k$ identity matrix in the top left corner with zeros everywhere else.

(e) It follows that $\text{trace}(P^{-1}AP) = k$, but also $\text{trace}(P^{-1}(AP)) = \text{trace}((AP)P^{-1}) = \text{trace}(A)$, so $\text{trace}(A) = k$.

(f) Now suppose that $\text{trace}(A) = 1$, so $k = 1$, so $u_1 \in U$ and $u_k \in V$ for $k > 1$. Now $q(u_1)u_i = \langle u_1, u_i\rangle u_1$. If $i > 1$ we have $u_1 \in U$ and $u_i \in V$ so $\langle u_1, u_i\rangle = 0$ and thus $q(u_1)u_i = 0 = Au_i$. If $i = 1$ then $\langle u_1, u_1\rangle = 1$ (as u_1 was chosen to be a unit vector) and so $q(u_1)u_1 = u_1 = Au_1$. As the elements u_i form a basis and $Au_i = q(u_1)u_i$ for all i, it follows that $q(u_1) = A$ as claimed.

Q2:

(a)



(b),(c) Take $U = F = \mathbb{R}$, which is both open and closed in $\mathbb{R}$. From the graph (and the observation that $f(x)$ is never zero) we see that $f(U) = f(F) = (0, 1]$, which is neither open nor closed.

(d) Take $K = [0, 1]$, which is certainly compact. We have $0 \leq f(x) \leq 1$ for all $x \in \mathbb{R}$, so $f^{-1}(K)$ is all of $\mathbb{R}$, which is not compact.

(e) Take $A = (-\infty, 0)$ and $B = (0, \infty)$. Then $f(A) = f(B) = (0, 1)$, so $f(A) \cap f(B) = (0, 1)$. On the other hand $A \cap B = \emptyset$ so $f(A \cap B) = \emptyset \neq f(A) \cap f(B)$.

Q3:

(a) Let W be an open subset of Y. As $f\colon U \to Y$ is continuous, we see that $f^{-1}W$ is open in U. As U is open in X it follows that $f^{-1}W$ is poen in X. Similarly, $g^{-1}W$ is open in V and hence in X. It is easy to see that $h^{-1}W = f^{-1}W \cup g^{-1}W$, so $h^{-1}W$ is open. This proves that h is continuous as claimed.

(b) Take $X = \mathbb{R}$ and $U = (-\infty, 0)$ and $F = [0, \infty)$. Define $f: U \to \mathbb{R}$ by $f(x) = 0$ for all x; this is constant and thus certainly continuous. Define $g: F \to \mathbb{R}$ by $g(x) = 1$ for all x, which is again continuous. The compatibility condition is automatically satisfied because $U \cap F = \emptyset$, so there is nothing to check. The resulting map $h: \mathbb{R} \to \mathbb{R}$ is clearly discontinuous at $x = 0$.

Q4:

(a) Put $A = f(x, y, z)$. Then we have

$$\text{trace}(A) = \frac{1}{2}((1 + x - yz) + (1 - x + yz)) = 1,$$

and $x^2 + y^2 = 1$ so

$$\begin{aligned} 4\det(A) &= (1 + x - yz)(1 - x + yz) - (y - z + xz)(y + z + xz) \\ &= (1 - x^2 - y^2z^2 + 2xyz) - (y^2 + 2xyz + x^2z^2 - z^2) \\ &= (1 - x^2 - y^2)(1 + z^2) = 0, \end{aligned}$$

so $\det(A) = 0$.

(b) Substituting this into the Cayley-Hamilton theorem gives $A^2 - 1.A + 0.I = 0$, or equivalently $A^2 = A$. We also have $\text{trace}(A) = 1$, so $A \in X$ as required.

(c) Put $c = \cos(\theta)$ and $s = \sin(\theta)$ and $A = R_\theta P(z) R_\theta^{-1}$. We then have

$$\begin{aligned} A &= \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \begin{pmatrix} 1 & z \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \\ &= \begin{pmatrix} c & cz \\ s & sz \end{pmatrix} \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \\ &= \begin{pmatrix} c^2 - scz & sc + c^2z \\ sc - s^2z & s^2 + scz \end{pmatrix} \end{aligned}$$

Now put $S = \sin(2\theta) = 2sc$ and $C = \cos(2\theta) = c^2 - s^2$. Using these formulae and the relation $s^2 + c^2 = 1$ we obtain

$$\begin{aligned} s^2 &= (1 - C)/2 \\ c^2 &= (1 + C)/2 \\ sc &= S/2, \end{aligned}$$

so

$$A = \frac{1}{2} \begin{pmatrix} 1 + C - Sz & S + z + Cz \\ S - z + Cz & 1 - C + Sz \end{pmatrix} = f(C, S, z)$$

as claimed.

(d) Suppose that $B \in X$ and $Ae_1 = e_1$. If $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then

$$Be_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}$$

so we must have $a = 1$ and $c = 0$. We also have $a + d = \text{trace}(B) = 1$, so we must have $d = 0$. This means that $B = \begin{pmatrix} 1 & c \\ 0 & 0 \end{pmatrix} = P(c)$.

(e) Suppose that $A \in X$. Then $\text{trace}(A) = 1$ so $A \neq 0$ so we can find $v \in \mathbb{R}^2$ with $Av \neq 0$. Put $u = Av/\|Av\|$. Then u is a unit vector, and $Au = A^2v/\|Av\| = Av/\|Av\| = u$ (because $A^2 = A$). Because u is a unit vector, we have $u = (\cos(\theta), \sin(\theta)) = R_\theta e_1$ for some θ. Put $B = R_\theta^{-1} A R_\theta$; it is easy to see that this again lies in X. Moreover, $Be_1 = R_\theta^{-1} Au = R_\theta^{-1} u = e_1$. It follows from (d) that $B = P(z)$ for some z, and thus that $A = R_\theta P(z) R_\theta^{-1} = f(\cos(2\theta), \sin(2\theta), z)$. This shows that f is surjective.

(f) Suppose that $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = f(x,y,z)$, so that

$$\begin{aligned} a &= (1+x-yz)/2 \\ b &= (y+z+xz)/2 \\ c &= (y-z+xz)/2 \\ d &= (1-x+yz)/2. \end{aligned}$$

We see immediately that $b-c=z$ and $b+c=y+xz$ so $b^2-c^2=(b-c)(b+c)=yz+xz^2$ so $b^2-c^2+2a-1=x(1+z^2)$ so $(b^2-c^2+2a-1)/(1+(b-c)^2)=x$. Similarly, we have $2(b-a(b-c))=(y+z+xz)-(1+x-yz)z=y(1+z^2)$ and so $2(b-a(b-c))/(1+(b-c)^2)=y$. It follows that $g(f(x,y,z))=g(A)=(x,y,z)$ as claimed. Thus, if $f(x',y',z')=f(x,y,z)$ we can apply g to deduce that $(x',y',z')=(x,y,z)$. This means that f is injective, and hence (by (e)) a bijection. As f is a bijection and $g\circ f=1$ we must have $f^{-1}=g$. The formula for g shows that f^{-1} is continuous, so f is a homeomorphism.

Q5:

(a) We have $\det(R(x,y))=x^2+y^2=1$ and $\det(D(a))=a.(1/a)=1$ and $\det(T(b))=1$ so $R(x,y)$, $D(a)$ and $T(b)$ all lie in $SL_2\mathbb{R}$.

(b) Suppose that $B=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then

$$Be_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}.$$

Thus, if $Be_1=e_1$ we must have $a=1$ and $c=0$. This means that $\det(B)=ad-bc=d$. Thus, if $B\in SL_2\mathbb{R}$ we must have $d=1$ and thus $B=\begin{pmatrix} 1 & b \\ 0 & d \end{pmatrix}=T(b)$.

(c) Suppose that $A\in SL_2\mathbb{R}$. Put $u=Ae_1$ and $a=\|u\|$ and $v=u/a$, so $v\in S^1$ and $Ae_1=av$. Note also that $R(v)e_1=v$, so $R(v)^{-1}Ae_1=aR(v)^{-1}v=ae_1$. It is also easy to see that $D(a)^{-1}ae_1=e_1$, so the matrix $B=D(a)^{-1}R(v)^{-1}A$ satisfies $Be_1=e_1$. We also have $\det(B)=\det(D(a))^{-1}\det(R(v))^{-1}\det(A)=1$ so $B\in SL_2\mathbb{R}$. Thus (b) tells us that $B=T(b)$ for some b.

(d) We now multiply the equation $D(a)^{-1}R(v)^{-1}A=T(b)$ by $R(v)D(a)$ to see that $A=R(v)D(a)B=R(v)D(a)T(b)=f(v,a,b)$. This shows that f is surjective.

(e) Put $A=f(u,a,b)=R(u)D(a)T(b)$ (where as usual $\|u\|=1$ and $a>0$). It is easy to see that $T(b)e_1=e_1$ and $D(a)e_1=ae_1$ and $R(u)ae_1=aR(u)e_1=au$, so $Ae_1=au$. This means that $\|Ae_1\|=a\|u\|=a$ and $Ae_1/\|Ae_1\|=(au)/a=u$. Thus $g(f(u,a,b))=g(A)=(u,a)$ as claimed.

(f) Suppose that $f(u,a,b)=f(u',a',b')$. Then $g(f(u,a,b))=g(f(u',a',b'))$, or in other words $(u,a)=(u',a')$, so $u=u'$ and $a=a'$. We are given that $f(u,a,b)=f(u',a',b')$, or in other words $f(u,a,b)=f(u,a,b')$, so $R(u)D(a)T(b)=R(u)D(a)T(b')$. We can multiply this equation by $D(a)^{-1}R(u)^{-1}$ to see that $T(b)=T(b')$, which clearly means that $b=b'$, so $(u,a,b)=(u',a',b')$. This shows that f is injective, and thus bijective.

(g) Suppose that $A_n\to A$ in $SL_2\mathbb{R}$, and put $f^{-1}(A_n)=(u_n,a_n,b_n)$ and $f^{-1}(A)=(u,a,b)$. This means that $A_n=f(u_n,a_n,b_n)$, and $A=f(u,a,b)$, so $g(A_n)=(u_n,a_n)$ and $g(A)=(u,a)$. The map g is clearly continuous and $A_n\to A$ so $g(A_n)\to g(A)$ so $u_n\to u$ and $a_n\to a$. Now put $B_n=D(a_n)^{-1}R(u_n)^{-1}$ and $B=D(a)^{-1}R(u)^{-1}$; it follows from the above that $B_n\to B$, and thus that $B_nA_n\to BA$. Moreover, $A_n=f(u_n,a_n,b_n)=R(u_n)D(a_n)T(b_n)$ so $B_nA_n=T(b_n)$, and similarly $BA=T(b)$. We thus see that $T(b_n)\to T(b)$, and it follows easily that $b_n\to b$. This means that $(u_n,a_n,b_n)\to(u,a,b)$, or in other words $f^{-1}(A_n)\to f^{-1}(A)$. This shows that f^{-1} is continuous, so f is a homeomorphism as claimed.