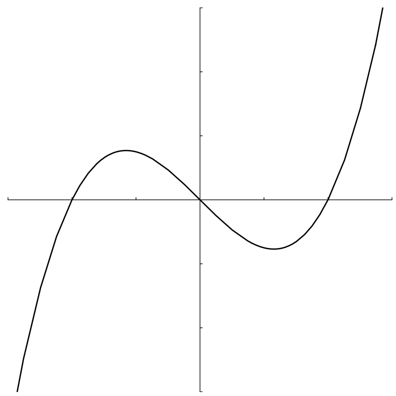
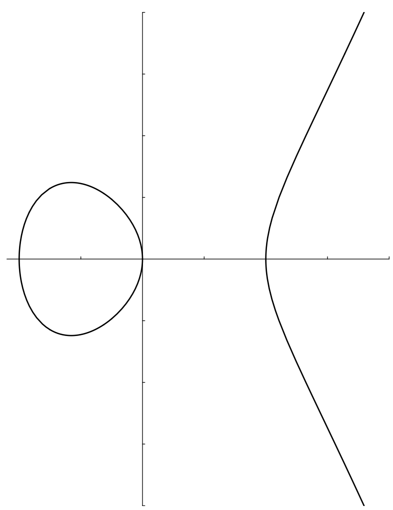
Algebraic Topology Problem Set 4 — Solutions

Q1: The graph of $x^3 - x$ is as follows:



Note that when $x < -1$ or $0 < x < 1$ we have $x^3 - x < 0$ so there is no solution to the equation $y^2 = x^3 - x$. When $x = 0$ or $x = \pm 1$ then $x^3 - x = 0$ and the unique solution is $y = 0$. When $-1 < x < 0$ or $x > 1$ then $x^3 - x > 0$ and there are two solutions, given by $y = \pm\sqrt{x^3 - x}$. We end up with the following picture:



If $(x, y) \in X$ then $x^3 - x = y^2 \geq 0$ but $(1/2)^3 - (1/2) < 0$ so $x \neq 1/2$ so $2x - 1 \neq 0$. Thus, the function f is nonzero everywhere on X. The points $a = (0, 0)$ and $b = (1, 0)$ lie in X and $f(a) < 0$ and $f(b) > 0$ so $a \not\sim b$ so X is not path-connected. (Of course, this is visually obvious from the picture, but the method can be generalised to spaces that are less easy to visualise.)

Q2:

(a),(b) Consider the continuous function $f \colon \mathbb{C} \to \mathbb{C}$ given by $f(x) = \exp(ix)$. We have $f(x)f(-x) = \exp(0) = 1$ so $f(x) \neq 0$, so we can regard f as a map $\mathbb{C} \to \mathbb{C} \setminus \{0\}$. If $z \in \mathbb{C} \setminus \{0\}$ then we can write $z = re^{i\theta}$ for some $r > 0$ and $\theta \in \mathbb{R}$. As r is strictly positive, $\log(r)$ is defined and we have $z = \exp(\log(r) + i\theta) = f(\theta - ir)$. This shows that f is a continuous surjection from $\mathbb{C}$ to $\mathbb{C} \setminus \{0\}$. It is easy to see that $|f(x + iy)| = |e^{ix-y}| = e^{-y}|e^{ix}| = e^{-y}$ and to deduce that f also gives a continuous surjection from $\mathbb{R}$ to S^1.

(c) As $\mathbb{R}$ is connected and $\mathbb{R} \setminus \{0\}$ is not, there can be no continuous surjection from $\mathbb{R}$ to $\mathbb{R} \setminus \{0\}$. Alternatively, we can argue as follows. Let $f \colon \mathbb{R} \to \mathbb{R} \setminus \{0\}$ be continuous and surjective. Then (by surjectivity) we can find $a, b \in \mathbb{R}$ with $f(a) = -1$ and $f(b) = 1$. The intermediate value theorem tells us that there exists c between a and b such that $f(c) = 0$. This contradicts the fact that f is a map from $\mathbb{R}$ to $\mathbb{R} \setminus \{0\}$.

(d) S^2 is bounded and closed in $\mathbb{R}^3$, so it is compact. I claim that $S^2 \setminus \{N\}$ is not closed in $\mathbb{R}^3$, and so is not compact. Given this, there can be no continuous surjection from S^2 to $S^2 \setminus \{N\}$. To see that $S^2 \setminus \{N\}$ is not closed, recall that $N = (0, 0, 1)$ and put $a_k = (0, \sin(\pi/k), \cos(\pi/k))$. Then $a_k \in S^2 \setminus \{N\}$ and $a_k \to N$ and $N \notin S^2 \setminus \{N\}$; this means that $S^2 \setminus \{N\}$ is not closed.

(e) The map $f(x, y, z) = z$ gives a continuous surjection from S^2 to the interval $[-1, 1]$. The map $g(t) = (\cos(\pi t), \sin(\pi t))$ gives a continuous surjection from $[-1, 1]$ to S^1. Thus, the composition $gf : S^2 \to S^1$ is again a continuous surjection.

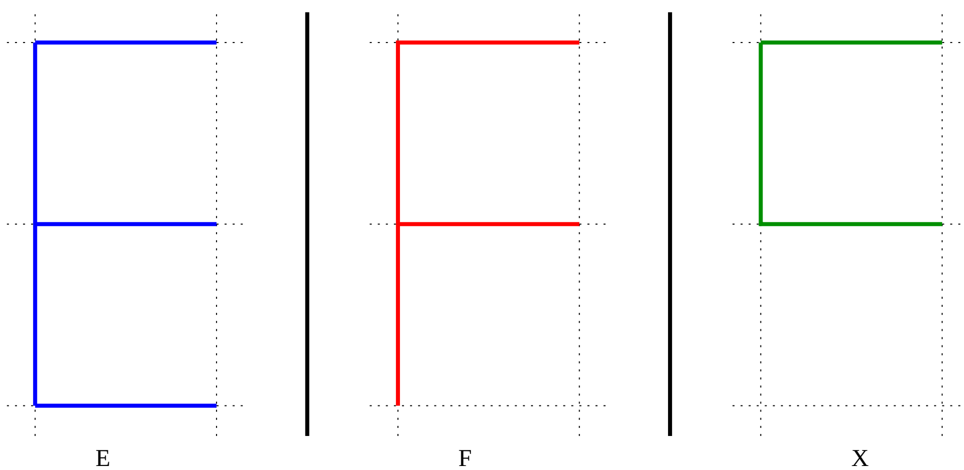
Q3:

(a) The invariants are as follows:

	A	B	C	D	E	F
$a(R)$	3	2	2	1	3	3
$b(R)$	1	2	1	2	1	1

(b) A, E and F have the same invariants. No other pair of different letters has the same invariants. It is possible to remove a single point from E to get a space with three components (and similarly for F) but this is not possible for A, so A is not homeomorphic to E or F. However, it is not hard to see that E is homeomorphic to F. To be horribly explicit, we could use the following formal definitions:

$$E = (\{0\} \times [0, 2]) \cup ([0, 1] \times \{0, 1, 2\}) \subset \mathbb{R}^2$$
$$F = (\{0\} \times [0, 2]) \cup ([0, 1] \times \{1, 2\}) \subset \mathbb{R}^2.$$

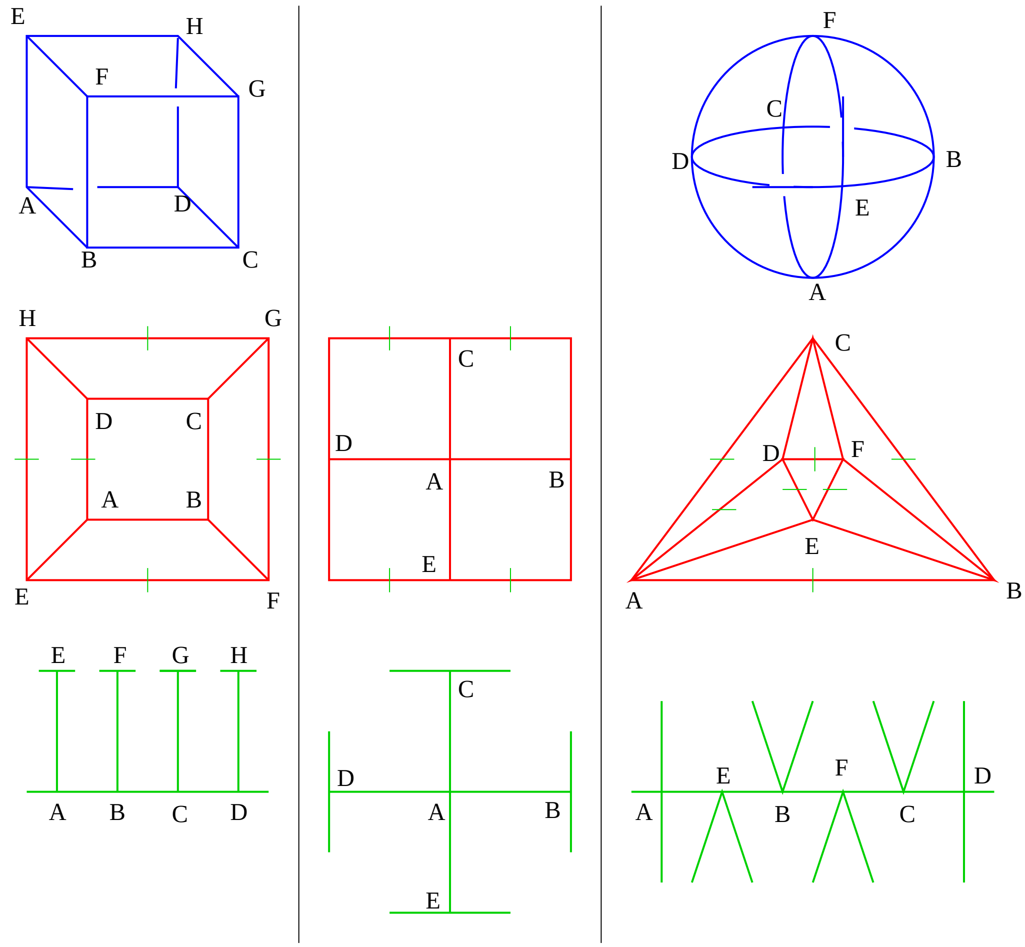


Let X be the union of the top two horizontal lines and the top half of the vertical line in E, so $X = (\{0\} \times [1, 2]) \cup ([0, 1] \times \{1, 2\})$. Define $f : E \to F$ by

$$f(x, y) = \begin{cases} (x, y) & \text{if } (x, y) \in X \\ (0, (y + 1/2)) & \text{if } (x, y) \in \{0\} \times [0, 1] \\ ((1 - x)/2, 0) & \text{if } (x, y) \in [0, 1] \times \{0\}. \end{cases}$$

(c) The conclusion is that E is homeomorphic to F, and no other pair of different letters are homeomorphic to each other.

Q4: The solution is illustrated by the following diagram:



In the left hand column we have a picture of X_0, and then a subset of $\mathbb{R}^2$ that is homeomorphic to X_0. On this diagram we have marked five cuts; after making these cuts, the remaining space is homeomorphic to the one illustrated at the bottom of the left hand column. From this it is clear that no more cuts can be made without disconnecting the space.

This does not absolutely prove that $a(X_0) = 5$; perhaps if we made a different choice for the first five cuts, there might be room to make a sixth one. In fact this does not happen; to prove this, one can either work through the various possibilities, or analyse how the Euler characteristic changes under cutting. We will not give the details here.

Similar arguments illustrated in the second and third columns show that $a(X_1) = 4$ and $a(X_2) = 7$. Thus $a(X_0)$, $a(X_1)$ and $a(X_2)$ are distinct, so no two of the spaces are homeomorphic to each other.

All that is left is to justify our picture of X_2. Note that $xyz = 0$ iff at least one of x, y and z is zero, iff (x, y, z) lies in the xy-plane, the xz-plane or the yz-plane. The intersection of each of these planes with S^2 is a circle, so X_2 is the union of three intersecting circles. The points of intersection are where two of the coordinates are zero, and so the third one is ± 1. From this it is easy to see that the picture is as shown.

Q5:

(a) In the case $\alpha = 0$ we need to show that β is the only eigenvalue of βI, which is trivial; so suppose that $\alpha \neq 0$.

As λ_i is an eigenvalue of A, we know that $A - \lambda_i I$ is not invertible, so $\alpha(A - \lambda_i I)$ is not invertible either. We have

$$(\alpha A + \beta I) - (\alpha\lambda_i + \beta)I = \alpha(A - \lambda_i I),$$

and this is non-invertible, so $(\alpha\lambda_i + \beta)$ is an eigenvalue of $\alpha A + \beta I$. Conversely, suppose that κ is an eigenvalue of $\alpha A + \beta I$, which means that $\alpha A + (\beta - \kappa)I$ is not invertible, so $\alpha^{-1}(\alpha A + (\beta - \kappa)I)$ is not invertible either. However, this matrix is just $A - (\kappa - \beta)\alpha^{-1}I$, so we see that $(\kappa - \beta)\alpha^{-1}$ is an eigenvalue of A, say $(\kappa - \beta)\alpha^{-1} = \lambda_i$ for some i. This can be rearranged to give $\kappa = \alpha\lambda_i + \beta$, as required.

(b) This is geometrically clear: if 0 lies on the line segment between λ_i and μ, then μ must be a negative real multiple of λ_i. Algebraically, if $0 = (1-t)\lambda_i + t\mu$ fro some $t \in [0,1]$ then in fact $t \in (0,1)$ (because $\lambda_i, \mu \neq 0$) and so $(t^{-1} - 1) \in (0, \infty)$ and $\mu = -(t^{-1} - 1)\lambda_i \in L_i$. Conversely, if $\mu \in L_i$ then $\mu = -s\lambda_i$ for some $s > 0$. If we put $t = (1+s)^{-1}$ then $(1-t)\lambda_i + t\mu = 0$ as required.

(c) Let C be the linear path from A to μI in $M_n\mathbb{C}$, so that $C(t) = (1-t)A + tI$. By part (a), the eigenvalues of $C(t)$ are the numbers $(1-t)\lambda_i + t\mu$; in other words, as t increases from 0 to 1 the i'th eigenvalue of $C(t)$ moves along the linear path from λ_i to μ. The path is contained in $GL_n(\mathbb{C})$ iff $C(t)$ is invertible for all $t \in I$, iff none of these eigenvalues is zero for any $t \in I$. By part (b), this holds iff $\mu \notin L_i$ for all i.

(d) Just choose a point $\mu \in \mathbb{C}$ not lying in any of the lines $L_1, \ldots, L_r$; the linear path joins A to μI in $GL_n(\mathbb{C})$.

(e) Suppose we have some $\mu \in \mathbb{C}$ with $\mu \neq 0$, say $\mu = re^{i\theta}$. Define $D(t) = (r - tr + t)e^{i(1-t)\theta}I$; this gives a path from μI to I in $GL_n(\mathbb{C})$.

(f) For any $A \in GL_n(\mathbb{C})$, part (d) shows that $A \sim \mu I$ for some μ, and part (e) gives $\mu I \sim I$, so $A \sim I$. If B is another element of $GL_n(\mathbb{C})$ then $B \sim I$ by the same argument, so $A \sim B$.