Algebraic Topology Problem Set 5 — Solutions

Q1: Put $g(x) = f(x)/|f(x)|$; as $f(x)$ is never zero, this gives a well-defined, continuous map $g\colon X \longrightarrow \mathbb{R}$. If $f(x) > 0$ then $g(x) = 1$, and if $f(x) < 0$ then $g(x) = -1$; thus $g(x)^2 = 1$ for all x. Next, if $t \in I$ and $x \in X$ we certainly have $(1-t) \geq 0$ and $t|f(x)| \geq 0$. The first of these is only zero when $t = 1$, and the second is only zero when $t = 0$, so the sum is never zero. We can thus define a continuous map $h\colon I \times X \longrightarrow \mathbb{R}$ by $h(t, x) = f(x)/((1-t) + t|f(x)|)$. This is evidently a homotopy between f and g.

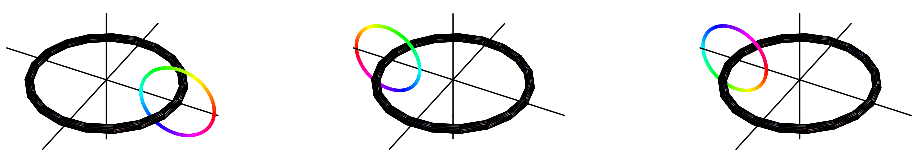
Q2:

(a) Define $h\colon I \times I \longrightarrow X$ by $h(s, t) = u(st)$. Then $h(1, t) = u(t)$ and $h(0, t) = u(0) = c_{u(0)}(t)$ so h is a homotopy between u and $c_{u(0)}$.

(b) We have seen that $u \simeq c_{u(0)}$ and $v \simeq c_{v(0)}$ so it will suffice to show that $c_{u(0)} \simeq c_{v(0)}$. As X is path-connected, we can choose a path w from $u(0)$ to $v(0)$. We can then define $k\colon I \times I \longrightarrow X$ by $k(s, t) = w(s)$. We have $k(0, t) = w(0) = u(0) = c_{u(0)}(t)$ and $k(1, t) = w(1) = v(0) = c_{v(0)}(t)$, so k is the required homotopy.

Q3:

(a) The three pictures show u, v and w.



(b) Just use the linear homotopy: $h(t, x, y) = (1-t)u(x, y) + tv(x, y)$.

(c) Suppose that $(x, y) \in S^1$. If $u(x, y) \in S^1$ then we must have $y/2 = 0$ and $(x/2+1)^2 + 0^2 = 1$. The first of these means that $y = 0$, and $(x, y) \in S^1$ so $x = \pm 1$. This is inconsistent with the equation $(x/2+1)^2 + 0^2 = 1$, so we cannot have $u(x, y) \in S^1$ after all, so we must have $u(x, y) \in \mathbb{R}^3 \setminus S^1 = X$. Essentially the same argument works for v and w.

(d) Define $k\colon I \times S^1 \longrightarrow \mathbb{R}^3$ by

$$\begin{aligned} k(t, x, y) &= \text{ rotation of } u(x, y) \text{ through angle } \pi t \text{ around the } z\text{-axis} \\ &= (\cos(\pi t)(x/2+1), \sin(\pi t)(x/2+1), y/2). \end{aligned}$$

If $k(t, x, y) \in S^1$ then we must have $y/2 = 0$ and $(\cos(\pi t)(x/2+1))^2 + (\sin(\pi t)(x/2+1))^2 = 1$. These equations reduce easily to $y = 0$ and $(x/2+1)^2 = 1$ so $x = 0$ or $x = -4$. We also have $x^2 + y^2 = 1$ because $(x, y) \in S^1$. This is clearly inconsistent, so we must have $k(t, x, y) \notin S^1$, so k is a map from $I \times S^1$ to X.

As $(\cos(0), \sin(0)) = (1, 0)$ we have $k(0, x, y) = u(x, y)$. As $(\cos(\pi), \sin(\pi)) = (-1, 0)$ we have $k(1, x, y) = v(x, y)$, so k is the required homotopy between u and v.

(e) Define

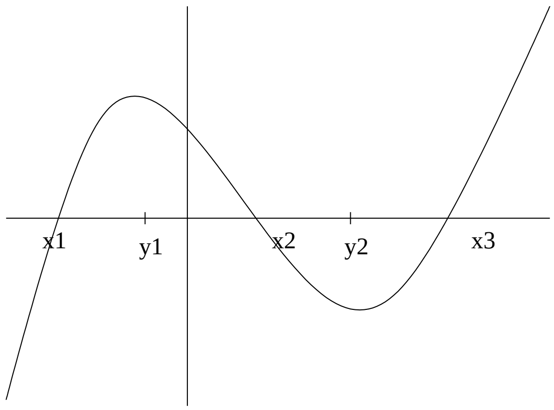
$$\begin{aligned} m(t, x, y) &= v(\text{ rotation of } (x, y) \text{ through } \pi t\ ) \\ &= v(\cos(\pi t)x - \sin(\pi t)y, \sin(\pi t)x + \cos(\pi t)y). \end{aligned}$$

As $m(t, x, y) = v(\text{something})$ we have $m(I \times S^1) \subseteq v(S^1)$. In particular, we have $m(I \times S^1) \subseteq X$, so $m\colon I \times S^1 \longrightarrow X$. We also have $m(0, x, y) = v(x, y)$ and $m(1, x, y) = v(-x, -y) = w(x, y)$, so m is a homotopy between v and w. The equation $v(x, y) = m(0, x, y)$ also tells us that $v(S^1) \subseteq m(I \times S^1)$, so $m(I \times S^1) = v(S^1)$ as required.

Q4: Let the roots of f be $\{x_1, \dots, x_n\}$, listed in order so that $x_1 < x_2 < \dots < x_n$. Suppose $1 \leq i < n$. As $f(x_i) = 0 = f(x_{i+1})$, the Mean Value Theorem tells us that $f'(y_i) = 0$ for some y_i with $x_i < y_i < x_{i+1}$. This gives $n - 1$ roots $y_1, \dots, y_{n-1}$ of $f'(x)$. However, $f'(x)$ is a polynomial of degree $n - 1$, so it has at most $n - 1$ roots, so we must have found all of them. We thus have

$$X = (-\infty, x_1) \cup (x_1, x_2) \cup \dots \cup (x_{n-1}, x_n) \cup (x_n, +\infty)$$
$$Y = \{y_1, \dots, y_{n-1}\} \subset X.$$

Each of the finite intervals in X is homotopy equivalent to the single point of Y that it contains. The two infinite intervals are also contractible. Thus X is homotopy equivalent to Y with two extra points adjoined. More precisely, if we put $Y' = Y \cup \{x_1 - 1, x_n + 1\}$ then the inclusion map $Y' \longrightarrow X$ is a homotopy equivalence.

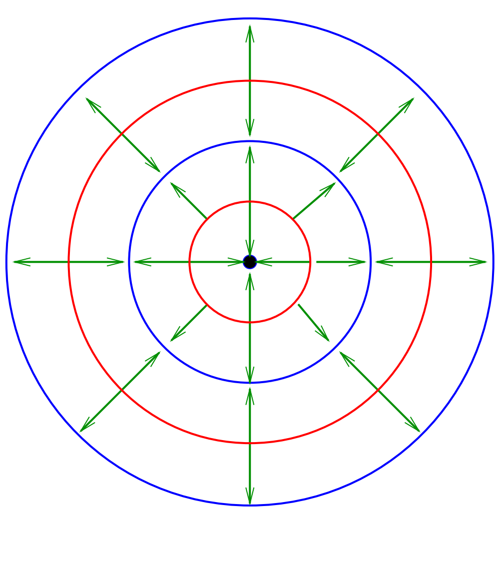


Q5: Define

$$U_+ = \{(x, y, z) \in S^2 \mid z > 0\}$$
$$U_- = \{(x, y, z) \in S^2 \mid z < 0\},$$

so $X = S^2 \setminus S^1 = U_+ \cup U_-$, and $U_+ \cap U_- = \emptyset$. If $x^2 + y^2 + z^2 = 1$ and $z \neq 0$ we must have $x^2 + y^2 < 1$ so $(x, y) \in \overset{\circ}{B}{}^2$. Define $f : U_+ \longrightarrow \overset{\circ}{B}{}^2$ by $f(x, y, z) = (x, y)$, and define $g : \overset{\circ}{B}{}^2 \longrightarrow U_+$ by $g(x, y) = (x, y, \sqrt{1 - x^2 - y^2})$. These maps show that U_+ is homeomorphic to $\overset{\circ}{B}{}^2$, which is contractible. Similarly, U_- is contractible. As X is the disjoint union of U_+ and U_-, it is homotopy equivalent to the 2-point space $\{N, S\}$.

Q6: First note that X is the set of points $x \in \mathbb{R}^2$ for which $r = \|x\|$ is not of the form $(n + \frac{1}{2})\pi$. Moreover, $\partial f / \partial r = \sin(r) = \sin(\|x\|)$, so Y is the set of points where $\|x\|$ has the form $n\pi$. Thus, in the picture below, X is the whole plane with the red circles removed, and Y consists of the central point together with the blue circles.



Let $f\colon Y \longrightarrow X$ be the inclusion map, and let $g\colon X \longrightarrow Y$ be the map indicated by the green arrows on the diagram. In other words, to get $g(x)$, we move inwards or outwards from x the least possible distance to get to a point of Y. If y already lies in Y then $g(y) = y$; in other words, we have $gf = 1_Y$. The line segment from x to $g(x) = fg(x)$ lies wholly within X, so we can define $h(t, x) = (1 - t)fg(x) + tx$ to get a homotopy from fg to 1_X.

We now give some formulae to make this more respectable. Write $S = \mathbb{R} \setminus \{n + \frac{1}{2} \mid n \in \mathbb{Z}\}$ and define $q\colon S \longrightarrow \mathbb{Z}$ by $q(s) =$ the nearest integer to s. It is not hard to see that this is continuous; the only problematic points are the points $n + \frac{1}{2}$, and we have excluded these from the domain. Moreover, if we let $p\colon \mathbb{Z} \longrightarrow S$ be the inclusion, then $qp = 1_{\mathbb{Z}}$. Now define $r\colon I \times S \longrightarrow \mathbb{R}$ by $r(t, s) = (1 - t)pq(s) + ts \in \mathbb{R}$. As $s \in S$ we have $n - \frac{1}{2} < s < n + \frac{1}{2}$ for some integer n and $q(s) = n$. It follows that $n - \frac{1}{2} < r(t, s) < n + \frac{1}{2}$ for all $t \in I$, and thus that $r(t, s) \in S$. We can thus think of r as a map $I \times S \longrightarrow S$, giving a homotopy $pq \simeq 1$.

For our formal definition of g, we put

$$g(x) = \begin{cases} q(\|x\|/\pi)\pi x/\|x\| & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Given that q is continuous, we see easily that g is continuous except possibly at $x = 0$. However, from the definition of q we see that $g(x) = 0$ whenever $\|x\| < \pi/2$, and using this we see that g is continuous at $x = 0$ as well.

Now, if $y \in Y \setminus \{0\}$ then $\|y\| = n\pi$ for some $n \in \mathbb{Z}$ so $q(\|y\|/\pi) = n$ so $g(y) = n\pi y/\|y\| = y$. Clearly we have $g(y) = y$ if $y = 0$ as well, and this shows that $gf = 1_Y$.

Next, define $h\colon I \times X \longrightarrow \mathbb{R}^2$ by

$$\begin{aligned} h(t, x) &= (1 - t)fg(x) + tx \\ &= \left((1 - t)q\left(\frac{\|x\|}{\pi}\right) + t\frac{\|x\|}{\pi} \right) \frac{\pi x}{\|x\|} \\ &= r(t, \|x\|/\pi)\pi x/\|x\|. \end{aligned}$$

The first formula shows that h is continuous. Because $r(t, s) \in S$ for all $t \in I$ and $s \in S$, the last formula shows that $h(t, x) \in X$ for all $t \in I$ and $x \in X$. (Strictly sepeaking, we need to consider $x = 0$ separately, but that is easy). This gives the required homotopy.

Q7:

(i) There are various possible ways to answer this question. One possibility is to say that x is an endpoint if there is no path-connected open set U such that $x \in U$ and $U \setminus \{x\}$ is disconnected. Suppose that x satisfies this definition and that $f\colon X \longrightarrow Y$ is a homeomorphism with $f(x) = y$ say. If V is a path-connected open subset of Y containing y then the set $U = f^{-1}(V)$ is an open subset of X containing x. It is homeomorphic to V and thus path-connected. By our definition of endpoints, the set $U \setminus \{x\}$ must be path-connected as well. Moreover, f gives a homeomorphism $U \setminus \{x\} \longrightarrow V \setminus \{y\}$, so $V \setminus \{y\}$ is path-connected. This proves that y is an endpoint of Y, as required.

Another possibility is to say that x is an endpoint of X if there is an open set $U \subset X$ such that $x \in U$ and a homeomorphism $u\colon U \longrightarrow [0, 1)$ such that $u(x) = 0$. With any reasonable definition of "one-dimensional objects", the two definitions will give the same answer. However, according to the first definition 0 is an endpoint of $\mathbb{R}^2$, but that is not the case according to the second definition.

(ii) In the spirit of our first definition above, we could say that x is a multiple point of X if there is a path-connected open set U containing x such that $U \setminus \{x\}$ has at least three path-components.

In the spirit of our second definition, we could define

$$W_n = \{re^{2\pi ik/n} \mid 0 \leq r < 1 \text{ and } k \in \mathbb{Z}\} = \{z \in \mathbb{C} \mid z^n \in [0, 1)\},$$

so W_n consists of n line segments meeting at the origin. We could say that x is a multiple point if there is an open set U containing x and a homeomorphism $f\colon U \longrightarrow W_n$ with $f(x) = 0$ for some $n > 2$.