# Algebraic Topology Problem Set 6 — Solutions

**Q1:** First note that $g_*f_* = (gf)_* = 1_{X*} = 1_{\pi_0 X}$, so $g_*f_*(u) = u$ for all elements $u \in \pi_0 X$.

(a) Suppose that $f_*(u) = f_*(v)$. Then $g_*f_*(u) = g_*f_*(v)$, or in other words $u = v$. This shows that $f_*$ is injective.

(b) Suppose that $u \in \pi_0 X$. If we put $v = f_*(u) \in \pi_0 Y$, we find that $g_*(v) = g_*f_*(u) = u$. As $u = g_*(v)$ we see that $u$ is in the image of $g$. This holds for all $u$, so $g_*$ is surjective.

(c) Suppose that $A = \{a_1, \dots, a_n\}$. (We assume that this lists all the elements without repetition, so $a_i \neq a_j$ when $i \neq j$, and $|A| = n$.) Suppose that $j\colon A \longrightarrow B$ is injective. Then $j(a_1), \dots, j(a_n)$ are all elements of $B$, and they are all different because $j$ is injective. Thus $B$ has at least $n$ different elements, ie $|B| \geq n$. Suppose instead that $q\colon B \longrightarrow A$ is surjective. Each element $a_i$ must lie in the image of $q$, so $a_i = q(b_i)$ for some $b_i \in B$. If $b_i = b_j$ then we must have $q(b_i) = q(b_j)$, or in other words $a_i = a_j$, but this only happens when $i = j$. This shows that $b_1, \dots, b_n$ are *distinct* elements of $B$, so again $|B| \geq n$.

**Q2:**

(a) Put $u(t) = h(t, x)$ and $v(t) = h(t, y)$, so $u$ is a path from $x$ to $a$, and $v$ is a path from $y$ to $a$. Then put $w = u * \overline{v}$, or equivalently

$$w(t) = \begin{cases} h(2t, x) & \text{if } 0 \leq t \leq 1/2 \\ h(2-2t, y) & \text{if } 1/2 \leq t \leq 1 \end{cases}$$

This gives a path from $x$ to $y$.

(b) First put $b = f(a)$ and $h'(t, y) = f(h(t, g(y)))$. Then $h'(0, y) = f(h(0, g(y))) = f(g(y))$ and $h'(1, y) = f(h(1, g(y))) = f(a) = b$ for all $y$. In other words, $h'$ is a homotopy between $fg$ and the constant map $c_b$ with value $b$. We are also given a homotopy $k$ from $1_Y$ to $fg$, and we can combine these to get a homotopy $m\colon I \times Y \longrightarrow Y$ between $1_Y$ and $c_b$. Explicitly, the formula is

$$m(t, y) = \begin{cases} k(2t, y) & \text{if } 0 \leq t \leq 1/2 \\ h'(2t-1, y) & \text{if } 1/2 \leq t \leq 1. \end{cases}$$

**Q3:**

(a) Suppose that $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then

$$AM = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} b & a \\ d & c \end{pmatrix}$$
$$MA = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ a & b \end{pmatrix}$$

Thus $AM = MA$ if and only if $c = b$ and $d = a$, so $A$ has the form $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$. If so then $\det(A) = a^2 - b^2 = (a+b)(a-b)$.

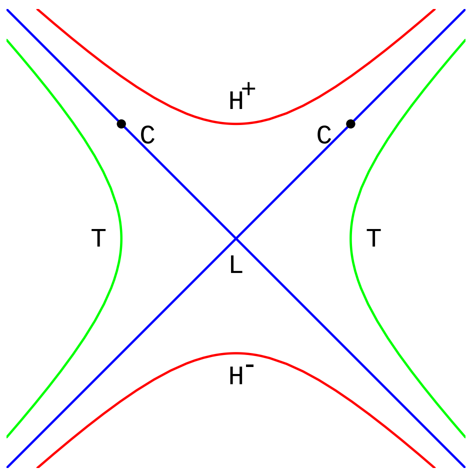
(b) Put $X = \{A \in GL_2\mathbb{C} \mid AM = MA\}$. If $A \in X$ then $A$ must have the form $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$ with $(a+b)(a-b) = \det(A) \neq 0$, so $a+b$ and $a-b$ are nonzero. We can thus define a map $f\colon X \longrightarrow \mathbb{C} \setminus \{0\} \times \mathbb{C} \setminus \{0\}$ by $f(A) = (a+b, a-b)$. Note that if $u = a+b$ and $v = a-b$ then $a = (u+v)/2$ and $b = (u-v)/2$. It follows that $f$ is a homeomorphism, with inverse

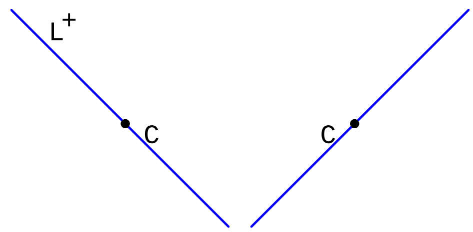
$$f^{-1}(u, v) = \frac{1}{2} \begin{pmatrix} u+v & u-v \\ u-v & u+v \end{pmatrix}.$$

(c) As $\mathbb{C} \setminus \{0\}$ is homotopy equivalent to $S^1$, we deduce that $X$ is homotopy equivalent to $S^1 \times S^1$, which is the torus.

**Q4:**

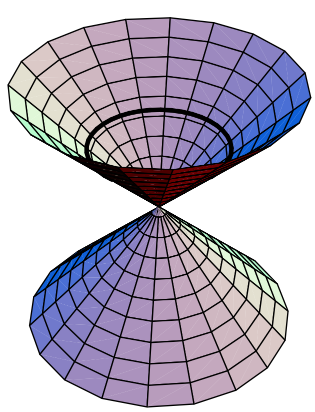
(a) The pictures are as follows:

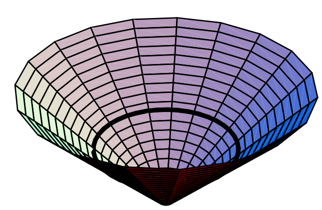



The space $L$ is star-shaped around $(0,0)$ and thus is contractible. The space $L^+$ consists of two disjoint lines, each of which is homotopy equivalent to a point. Thus $L^+ \simeq S^0$. The space $C$ consists of the two points $(-1,1)$ and $(1,1)$ so it is homeomorphic to $S^0$, and hence homotopy equivalent to $S^0$. The spaces $H$ and $T$ each consist of two disjoint, contractible curves so also $H \simeq T \simeq S^0$. The space $H^+$ is contractible.

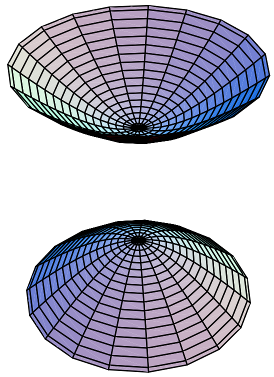
(b) The cone on the left is $L$, the half-cone on the right (which does not contain the origin) is $L^+$, and the black circle is $C$. For any point $a \in L$, the straight line joining $a$ to $0$ lies wholly in $L$, so $L$ is again star-shaped and contractible. As the vertex has been removed from $L^+$, we can straighten it out to get a cylinder like $S^1 \times (0,\infty)$, which is clearly homotopy equivalent to $S^1$. Also, $C$ is homeomorphic and thus homotopy equivalent to $S^1$.

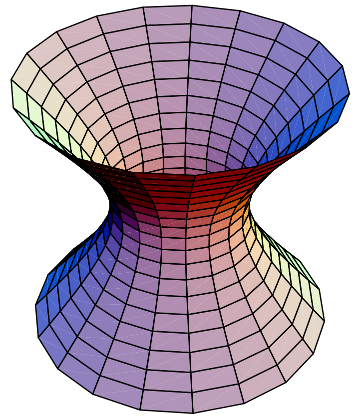




The picture on the left shows $H$, which consists of two disjoint surfaces, each one homeomorphic to $\mathbb{R}^2$ and thus contractible. It follows that $H \simeq S^0$ and $H^+$ is contractible. The picture on the right shows $T$. Here the walls can be straightened out to give a cylinder

$S^1 \times \mathbb{R}$, which is homotopy equivalent to $S^1$.





(c) (i) If $(t,x,y,z) \in L$ then $t^2 = x^2 + y^2 + z^2$ so $(st)^2 = (sx)^2 + (sy)^2 + (sz)^2$ (for any $s \in \mathbb{R}$), so $(st, sx, sy, sz) \in L$. This shows that $L$ is star shaped around the origin and thus contractible.

(ii) We can define $f\colon L^+ \longrightarrow (0,\infty) \times S^2$ by $f(t,x,y,z) = (t, x/t, y/t, z/t)$. This is a homeomorphism, with inverse $f^{-1}(t,x,y,z) = (t, tx, ty, tz)$. It follows that $L^+ \simeq S^2$.

(iii) $C = \{(1,x,y,z) \mid 1 = x^2 + y^2 + z^2\}$; this is clearly a copy of $S^2$.

(iv) Observe that

$$\begin{aligned} H^+ &= \{(t,x,y,z) \mid t^2 - x^2 - y^2 - z^2 = 1 \text{ and } t > 0\} \\ &= \{(t,x,y,z) \mid t^2 = 1 + x^2 + y^2 + z^2 \text{ and } t > 0\} \\ &= \{(t,x,y,z) \mid t = \sqrt{1 + x^2 + y^2 + z^2}\} \end{aligned}$$

We can thus define a homeomorphism $f\colon \mathbb{R}^3 \longrightarrow H^+$ by

$$f(x,y,z) = (\sqrt{1 + x^2 + y^2 + z^2}, x, y, z),$$

with inverse $f^{-1}(t,x,y,z) = (x,y,z)$. It follows that $H^+$ is contractible, and $H^-$ is also contractible by a similar proof. Thus $H = H^+ \amalg H^- \simeq S^0$.

(v) Observe that

$$\begin{aligned} T &= \{(t,x,y,z) \mid t^2 - x^2 - y^2 - z^2 = -1\} \\ &= \{(t,x,y,z) \mid x^2 + y^2 + z^2 = 1 + t^2\}. \end{aligned}$$

We can define a homeomorphism $\mathbb{R} \times S^2 \longrightarrow T$ by $f(t,x,y,z) = \sqrt{1+t^2}(x,y,z)$, with inverse $f^{-1}(t,x,y,z) = (t, (x,y,z)/\sqrt{1+t^2})$. It follows that $T \simeq S^2$.