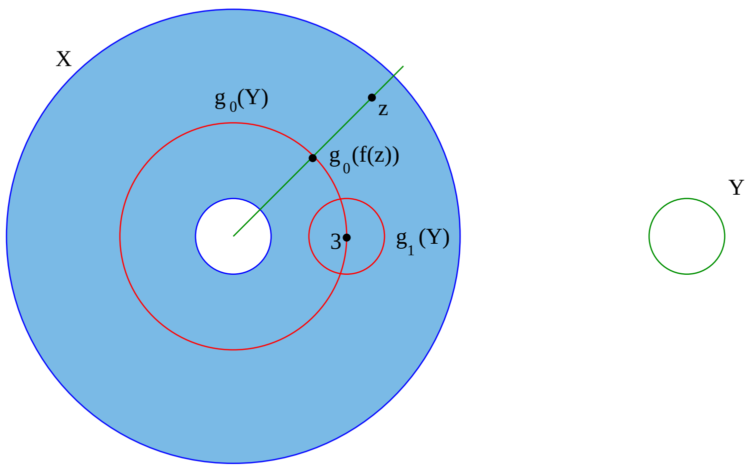
Algebraic Topology Problem Set 7 — Solutions

Q1: As f is linearly homotopic to g, for each point $x \in X$, the line segment from $f(x)$ to $g(x)$ in $\mathbb{R}^2$ must be wholly contained in the circle S^1. But the circle does not contain any line segments (other than points, which count as line segments of zero length, with the same start and end point). (Of course, the *disc* B^2 contains many line segments; but that is a different question altogether.) Thus, the line segment from $f(x)$ to $g(x)$ must have zero length, in other words $f(x) = g(x)$. We deduce that $f = g$.

Q2: The picture is:



(a) This is meaningless. The map f goes from X to Y, and g_0 goes from Y to X. It only makes sense to ask whether two maps are homotopic if they have the same source and target.
(b) Again meaningless, for the same reason: fg_0 goes from Y to Y, and 1_X goes from X to X.
(c) True. For $z \in Y = S^1$ we have $fg_0(z) = f(3z) = (3z)/|3z| = 3z/3 = z$, so fg_0 is equal to 1_Y, and thus certainly homotopic to 1_Y.
(d) True. We have $g_0 f(z) = g_0(z/|z|) = 3z/|z|$. This is a positive multiple of z and has $|g_0 f(z)| = 3$, so it is as indicated in the picture. The line segment from z to $g_0 f(z)$ lies wholly in X, so $g_0 f$ is linearly homotopic to 1_X.
(e) False. For every $z \in X$, we have $f(z) \in Y$, so $g_1 f(z)$ lies on the circle marked $g_1(Y)$. It follows that the line segment from $g_1 f(z)$ to 3 lies wholly in X, so $g_1 f$ is linearly homotopic to the constant map with value 3. However, we know that $\pi_1 X \simeq \mathbb{Z}$ so X is not contractible, so 1_X is not homotopic to a constant map. Thus $g_1 f \not\simeq 1_X$.

Q3: As X is contractible, there is a map $h\colon I \times X \longrightarrow X$ and a point $a \in X$ such that $h(0, x) = x$ and $h(1, x) = a$ for all $x \in X$.

(a) Define $m\colon I \times W \longrightarrow X$ by $m(t, w) = h(t, f(w))$. Then $m(0, w) = h(0, f(w)) = f(w)$ and $m(1, w) = h(1, f(w)) = a$, so m is a homotopy between f and the constant map with value a.
(b) Define $n\colon I \times X \longrightarrow Y$ by $n(t, x) = g(h(t, x))$. Then $n(0, x) = g(h(0, x)) = g(x)$ and $n(1, x) = g(h(1, x)) = g(a)$. Thus, n is a homotopy between g and the constant map with value $g(a)$.

Q4:

(a) Take $X = Y = S^1$ and $f(z) = z$ and $g(z) = -z$. The line segment from $f(z)$ to $g(z)$ runs through 0 and thus does not lie wholly in S^1. It follows that f is not linearly homotopic to g in Y. On the other hand, the map $h\colon I \times X \longrightarrow Y$ given by $h(t, z) = e^{i\pi t} z$ shows that f is (nonlinearly) homotopic to g.

(b) Take

$$\begin{aligned} X &= S^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2=1\} \\ Y &= \text{ the northern hemisphere } = \{(x,y,z) \in S^2 \mid z \geq 0\} \\ Z &= \text{ the southern hemisphere } = \{(x,y,z) \in S^2 \mid z \leq 0\}. \end{aligned}$$

Then X is connected, and each of Y and Z is homeomorphic to B^2 and thus is contractible. The intersection $Y \cap Z$ is just $\{(x,y,0) \mid x^2+y^2=1\}$, which is a copy of S^1, and thus is connected but not contractible.

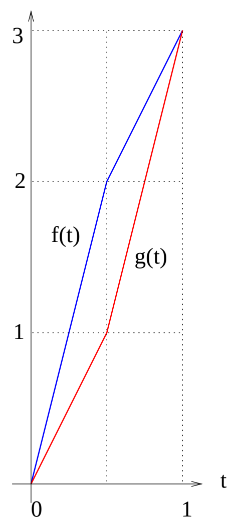
(c) Take $X = \mathbb{R} \setminus \{0\}$ and $Y = \mathbb{R}$, and define $f\colon X \longrightarrow Y$ by $f(x) = x$, so f is evidently injective. Then $\pi_0 X = \{-1, 1\}$ and $\pi_0 Y = \{0\}$, and $f_*\colon \pi_0 X \longrightarrow \pi_0 Y$ is given by $f_*(1) = f_*(-1) = 0$. This shows that f_* is not injective, but f itself is clearly injective.

(d) Define $f\colon \mathbb{R} \longrightarrow S^1$ by $f(t) = \exp(2\pi i t)$. This is clearly continuous and surjective. We also have $\pi_1 \mathbb{R} = 0$ and $\pi_1 S^1 = \mathbb{Z}$. The only homomorphism from $\{0\}$ to $\mathbb{Z}$ is zero, which is not surjective.

Q5: Put

$$f(t) = \begin{cases} 0 \leq t \leq 1/2 & 4t \\ 1/2 \leq t \leq 1 & 2t+1 \end{cases}$$

$$g(t) = \begin{cases} 0 \leq t \leq 1/2 & 2t \\ 1/2 \leq t \leq 1 & 4t-1 \end{cases}$$



For $t \in [0,1/4]$ we have $f(t) = 4t \in [0,1]$ and so $z(f(t)) = u(4t)$. For $t \in [1/4,1/2]$ we have $f(t) = 4t \in [1,2]$ and so $z(f(t)) = v(4t-1)$. For $t \in [1/2,1]$ we have $f(t) = 2t+1 \in [2,3]$ and so $z(f(t)) = w((2t+1)-2) = w(2t-1)$. From this we deduce directly that $z(f(t)) = ((u*v)*w)(t)$, or in other words $z \circ f = (u*v)*w$. A similar argument shows that $z \circ g = u*(v*w)$. Now define $h\colon I \times I \longrightarrow \mathbb{R}$ by $h(s,t) = (1-s)f(t) + sg(t)$. As $f(t), g(t) \in [0,3]$, it is easy to see that $h(s,t) \in [0,3]$, so we can regard h as a map $I \times I \longrightarrow [0,3]$. Note that $f(0) = g(0) = 0$, so $h(s,0) = 0$ for all s. Similarly $f(1) = g(1) = 3$, so $h(s,1) = 3$ for all s. Now define $k\colon I \times I \longrightarrow X$ by $k(s,t) = z(h(s,t))$. We find that $k(0,t) = z(h(0,t)) = z(f(t)) = ((u*v)*w)(t)$ and $k(1,t) = z(h(1,t)) = z(g(t)) = (u*(v*w))(t)$. We also see that $k(s,0) = z(0) = a$ and $k(s,1) = z(3) = d$ for all s. This shows that h is a homotopy rel endpoints from $(u*v)*w$ to $u*(v*w)$.