SPECTRAL SEQUENCES

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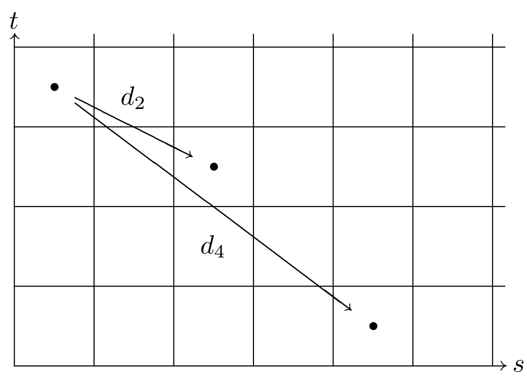
1. Introduction

Definition 1.1. A (multiplicative, first quadrant, cohomologically graded) spectral sequence consists of a sequence of "pages" E_r for $r \geq 2$. Each page is a bigraded Abelian group E_r^{st}, with $E_r^{st} = 0$ if $s < 0$ or $t < 0$. It come equipped with a differential

$$d_r \colon E_r^{st} \to E_r^{s+r,t-r+1}$$

satisfying $d_r^2 = 0$, and the next page E_{r+1} is the cohomology of E_r with respect to d_r:

$$E_{r+1}^{st} = \frac{\ker(d_r \colon E_r^{st} \to E_r^{s+r,t-r+1})}{\mathrm{image}(d_r \colon E_r^{s-r,t+r-1} \to E_r^{st})}.$$



Moreover, there are product maps $E_r^{st} \otimes E_r^{uv} \to E_r^{s+u,t+v}$ making E_r into a bigraded ring. It is commutative up to sign, and d_r is a derivation: if $a \in E_r^{st}$ and $b \in E_r^{uv}$ then

$$ab = (-1)^{(s+t)(u+v)} ba$$

$$d_r(ab) = d_r(a)b + (-1)^{s+t} a d_r(b).$$

Note that for fixed s and t, when r is sufficiently large the differential d_r starting at E_r^{st} ends below the s axis, and thus is zero; and the differential d_r ending at E_r^{st} starts to the left of the t axis, and thus is also zero. It follows that $E_r^{st} = E_{r+1}^{st}$ when $r \gg 0$. We write E_∞^{st} for this group.

We say that the spectral sequence *converges* to a graded ring A^* if there is a given filtration

$$A^u = F^0 A^u \geq F^1 A^u \geq \ldots \geq F^u A^u \geq F^{u+1} A^u = 0$$

such that $F^s A^u . F^t A^v \subseteq F^{s+t} A^{u+v}$, and given isomorphisms

$$F^s A^u / F^{s+1} A^u = E_\infty^{s,u-s}$$

that are compatible with the ring structures on E_∞ and A.

If so, note that there are *edge maps*

$$A^u \twoheadrightarrow F^0 A^u / F^1 A^u = E_\infty^{0,u} \rightarrowtail E_2^{0,u}$$

$$E_2^{u,0} \twoheadrightarrow E_\infty^{u,0} = F^u A^u \rightarrowtail A^u.$$

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2. The Serre spectral sequence

The main theorem is that for any fibration $F \to E \to B$, there are spectral sequences relating the (co)homology of F, E and B. We first give a theorem in which we make some restrictive assumptions to simplify the statement.

Theorem 2.1. *Let $F \to E \xrightarrow{q} B$ be a fibration, with B simply connected. Let K be a field, take all cohomology with coefficients in K, and assume that H^nB and H^nF are finite-dimensional for all n. Then there is a* Serre spectral sequence *with $E_2^{st} = H^s(B) \otimes_K H^t(F)$, which converges to the ring H^*E. (The last sentence is often written: there is a Serre spectral sequence $H^s(B) \otimes_K H^t(F) \Longrightarrow H^{s+t}E$.)*

We now give a more complicated statement which is more generally valid.

Theorem 2.2. *Let $q \colon E \to B$ be a fibration, and R a commutative ring. Then there is a Serre spectral sequence*

$$H^s(B; \mathcal{H}^t(F;R)) \Longrightarrow H^{s+t}(E;R),$$

where $\mathcal{H}^t(F;R)$ means the local coefficient system $b \mapsto H^t(q^{-1}\{b\};R)$. Similarly, if B' is a subspace of B and $E' = q^{-1}B'$ then there is a relative Serre spectral sequence

$$H^s(B, B'; \mathcal{H}^t(F;R)) \Longrightarrow H^{s+t}(E, E';R).$$

(This does not have a ring structure, but it does have a module structure over the previous spectral sequence.)

We will say nothing about the theory of local coefficient systems except to explain when they are unnecessary. Recall that there is a natural action of the H-group ΩB on the fibre $Pq \simeq F$. Using this, each element of $\pi_0\Omega B = \pi_1 B$ gives a homotopy class of maps $F \to F$, and thus a map $H^*(F;R) \to H^*(F;R)$. This construction gives an action of the group $\pi_1 B$ on H^*F.

Proposition 2.3. *If $F \to E \xrightarrow{q} B$ is a fibration, B is connected, and $\pi_1 B$ acts trivially on $H^*(F;R)$, then the E_2 terms of the above spectral sequences are just $H^s(B; H^t(F;R))$ and $H^s(B, B'; H^t(F;R))$.*

Proposition 2.4. *If E and B are H-groups, and $q \colon E \to B$ is both a fibration and an H-map, then $\pi_1 B$ acts trivially on F.*

We also have Serre spectral sequences in homology (as opposed to cohomology). We give another definition to summarise their properties.

Definition 2.5. A first quadrant homologically graded spectral sequence consists of pages E^r for $r \geq 2$, with $E^r_{st} = 0$ if $s < 0$ or $t < 0$. There are differentials $d^r \colon E^r_{st} \to E^r_{s+r,t-r+1}$ (the opposite direction to the cohomological case) with $(d^r)^2 = 0$ and $E^{r+1} = \ker(d^r)/\operatorname{image}(d^r)$. We say that such a spectral sequence converges to a graded group A_* if there is a filtration $0 = F_{-1}A_u \leq F_0 A_u \leq \ldots \leq F_u A_u = A_u$ and isomorphisms $F_s A_u / F_{s-1} A_u \simeq E^\infty_{s,u-s}$.

Theorem 2.6. *If $q \colon E \to B$ is a fibration and R is a ring then there is a homologically graded Serre spectral sequence*

$$H_s(B; \mathcal{H}_t(F;R)) \Longrightarrow H_{s+t}(E;R).$$

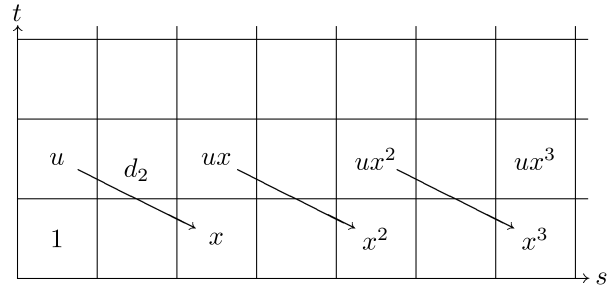
There is also a relative version.

3. Examples of the Serre spectral sequence

Example 3.1. Consider the fibration $S^1 \to S^{2n+1} \to \mathbb{C}P^n$. There is a Serre spectral sequence $H^*(\mathbb{C}P^n) \otimes H^*(S^1) \Longrightarrow H^*S^{2n+1}$. Recall that

$$\begin{aligned} H^*S^1 &= \mathbb{Z}[u]/u^2 && |u| = 1 \\ H^*S^{2n+1} &= \mathbb{Z}[v]/v^2 && |v| = 2n+1 \\ H^*\mathbb{C}P^n &= \mathbb{Z}[x]/x^{n+1} && |x| = 2. \end{aligned}$$

Thus $E_2 = \mathbb{Z}[u,x]/(u^2, x^{n+1})$ with $u \in E_2^{01}$ and $x \in E_2^{20}$. It turns out that $d_2(u) = x$ and $d_2(x) = 0$, and thus that $d_2(x^k u) = x^{k+1}$ and $d_2(x^k) = 0$. It follows that $E_3 = \mathbb{Z}\{1, ux^n\}$, and there are no more differentials, so $E_3 = E_\infty$. The filtration on $H^{2n+1}S^{2n+1}$ is given by $F^0 = \ldots = F^{2n} = \mathbb{Z}v$, and $F^{2n+1} = 0$, and the isomorphism $F^{2n}/F^{2n+1} = E_\infty^{2n,1}$ sends v to $\pm ux^n$. We illustrate the case $n = 3$:



Example 3.2. The above fibration can also be shifted to give a fibration

$$S^{2n+1} \to \mathbb{C}P^n \to \mathbb{C}P^\infty.$$

This gives a spectral sequence

$$\mathbb{Z}[x] \otimes E[u] = H^*\mathbb{C}P^\infty \otimes H^*S^{2n+1} \Longrightarrow H^*\mathbb{C}P^n = \mathbb{Z}[x]/x^{n+1},$$

with $x \in E_2^{2,0}$ and $u \in E_2^{0,2n+1}$. The differentials $d_2, \ldots, d_{2n+1}$ are all zero (there is nothing for them to hit). Then $d_{2n+2}(u) = x^{n+1}$ and $d_{2n+2}(x) = 0$ so $d_{2n+2}(x^i u) = x^{i+n+1}$. It follows that $E_{2n+3} = \mathbb{Z}[x]/x^{n+1}$ and there is no room for any more differentials.

Example 3.3. Given integers $k \leq l$, we can define the Milnor hypersurface

$$M = \{([z],[w]) \in \mathbb{C}P^k \times \mathbb{C}P^l \mid \sum_{i=0}^{k} z_i w_i = 0\}.$$

Recall that

$$H^*\mathbb{C}P^l = \mathbb{Z}[x]/x^{l+1}$$
$$H^*\mathbb{C}P^k = \mathbb{Z}[y]/y^{k+1}$$

There are obvious maps $p \colon M \to \mathbb{C}P^k$ and $q \colon M \to \mathbb{C}P^l$, defined by $p([z],[w]) = [z]$ and $q([z],[w]) = [w]$. We write x for $p^*x \in H^2M$ and y for $q^*y \in H^2M$. It turns out that

$$H^*M = \mathbb{Z}[x,y]/(x^{k+1}, y^l - y^{l-1}x + \ldots \pm x^l).$$

This has a basis $H^*M = \mathbb{Z}\{x^i y^j \mid i \leq k\ ,\ j < l\}$. One can show that there is a fibration $\mathbb{C}P^{l-1} \to M \xrightarrow{p} \mathbb{C}P^k$. The natural filtration of H^*M arising from this filtration is given by

$$F^{2s}H^*M = F^{2s-1}H^*M = \text{ ideal generated by } y^s.$$

The associated graded ring $G^*H^*M = \prod_s F^s/F^{s+1}$ is given by

$$G^*H^*M = \mathbb{Z}[\overline{x}, \overline{y}]/(\overline{x}^{k+1}, \overline{y}^l),$$

where $\overline{x} \in G^2$ is the image of x in F^2/F^3, and $\overline{y} \in G^0$ is the image of y in F^0/F^1. (Because $y^l = y^{l-1}x - \ldots \mp x^l \in F^1$, we have $\overline{y}^l = 0$ in the natural ring structure on G^*.) Thus, G^* has rather simpler structure than H^*M does.

There is a Serre spectral sequence $H^*(\mathbb{C}P^k) \otimes H^*(\mathbb{C}P^l) \Longrightarrow H^*M$. The E_2 page is just $\mathbb{Z}[x,z]/(x^{k+1}, z^l)$. Note that x and z both have even total degree (where the *total degree* of E_r^{st} is $s+t$) and that all differentials run from a slot of even total degree to one of odd total degree or *vice versa*, so all differentials are necessarily zero. It follows that $E_\infty = E_2$, and we have an obvious identification of E_∞ with G^*.

If we were using this spectral sequence to calculate H^*M, then we might be misled into believing that $y^l = 0$ in H^*M. This example shows that care is needed in deducing the multiplicative structure of the target ring H^*E from the E_∞ page of the spectral sequence.

We illustrate the case $k = l = 3$. We have written in the elements $d_r y^2$ to show that they all lie in slots where either s or t is odd, so they must all be zero.

y^2		xy^2		x^2y^2		x^3y^2
		d_2y^2				
y		xy	d_3y^2	x^2y		x^3y
				d_4y^2		
1		x		x^2	d_5y^2	x^3

Example 3.4. Consider the fibration $U(n-1) \xrightarrow{j} U(n) \xrightarrow{q} S^{2n-1}$, where $q(A) = Ae_n$. We will use this to calculate the cohomology of $U(n)$. We claim that there are canonically defined elements $a_{2k+1} \in H^{2k+1}U(n)$ for $0 \leq k < n$ such that

$$H^*U(n) = E[a_1, a_3, \ldots, a_{2n-1}].$$

To see this, assume the corresponding thing for $U(n-1)$ and consider the Serre spectral sequence $H^*S^{2n-1} \otimes H^*U(n-1) \Longrightarrow H^*U(n)$. Let u be the generator of $H^{2n-1}S^{2n-1}$, so the E_2 page is $E[a_{2i+1} \mid 0 \leq i < n-1] \otimes E[u]$, with $a_{2i+1} \in E_2^{0,2i+1}$ and $u \in E_2^{2n-1,0}$. The whole page is thus concentrated in the 0'th and $(2n-1)$'st columns, and the only possible differential is d_{2n-1}. For each $i < n-1$ we have $d_{2n-1}(a_{2i-1}) \in E_{2n-1}^{2n-1,2(i-n)}$, which lies below the axis $t = 0$ and thus is zero. It is even easier to see that $d_{dn-1}(b) = 0$. As d_{2n-1} is a derivation, we see that it vanishes on the whole algebra generated by the a_i's and b, which is the whole E_{2n-1} page, so $E_\infty = E_2$. This means that in the natural filtration of $H^*U(n)$, the quotient $F^0/F^1 = E_\infty^{0*}$ maps isomorphically by j^* to $H^*U(n-1)$, that $F^1 = F^2 = \ldots = F^{2n-1}$, that $F^{2n} = 0$, and that F^{2n-1} is a free module over F^0/F^1 on one generator q^*u.

For each $i < n-1$, the group $E_\infty^{0,2i+1}$ is the only nonzero term in total degree $2i+1$. It follows easily that there is a unique element $b_{2i+1} \in H^{2i+1}U(n)$ with $j^*b_{2i+1} = a_{2i+1}$. We also define $b_{2n-1} = q^*u$. All the b's lie in odd degrees, so they anticommute. We thus get a map $E[b_1, \ldots, b_{2n-1}] \to H^*U(n)$. The element b_{2n-1} lies in F^1, so we get a map $E[b_1, \ldots, b_{2n-3}] \to F^0/F^1$. Given that j^* induces an isomorphism $F^0/F^1 \to E[a_1, \ldots, a_{2n-3}]$ and $j^*b_{2i-1} = a_{2i-1}$, we conclude that our map $E[b_1, \ldots, b_{2n-3}] \to F^0/F^1$ is an isomorphism. As F^{2n-1} is a free module over F^0/F^1 on one generator b_{2n-1}, we conclude that our map $E[b_1, \ldots, b_{2n-1}] \to H^*U(n)$ is also an isomorphism.

We illustrate the case $n = 3$.

a_1a_3					a_1a_3u
a_3					a_3u
		d_2a_3			
a_1			d_3a_3		a_1u
1				d_4a_3	1

Example 3.5. Consider the fibration $\Omega S^{2n+1} \to PS^{2n+1} \to S^{2n+1}$. The path space PS^{2n+1} is contractible, so we have a Serre spectral sequence

$$E[u] \otimes H^* \Omega S^{2n+1} \Longrightarrow \mathbb{Z},$$

with $u \in E_2^{2n+1,0}$. As the E_2 page is concentrated in the columns $s = 0$ and $s = 2n+1$, the only possible differential is

$$d_{2n+1} \colon H^t(\Omega S^{2n+1}) \to H^{t-2n}(\Omega S^{2n+1})u.$$

As the spectral sequence converges to $\mathbb{Z}$, we see that the E_{2n+2} page must just be $\mathbb{Z}$, and so the above differential must be an isomorphism for $t > 0$. For $0 < t < 2n$ we know that $H^{t-2n}(\Omega S^{2n+1}) = 0$ so $H^t(\Omega S^{2n+1}) = 0$. Continuing inductively, we find that $H^{2nk}(\Omega S^{2n+1}) = \mathbb{Z}x_k$ for some x_k with $x_0 = 1$ and $d_{2n+1}(x_k) = x_{k-1}u$, whereas $H^t(\Omega S^{2n+1}) = 0$ for $t \neq 0 \pmod{2n}$.

We can now determine the multiplicative structure. We have $d_{2n+1}(x_k) = x_{k-1}u$, but the Leibniz rule gives $d_{2n+1}(x_1^k) = kx_1^{k-1}u$. It follows by induction that $x_1^k = k!x_k$, and thus that $x_j x_k = \binom{j+k}{j} x_{j+k}$.

Example 3.6. Consider the fibration $U(n) = \Omega BU(n) \to PBU(n) \to BU(n)$. This gives a Serre spectral sequence

$$E_2^{**} = \mathbb{Z}[c_1, \ldots, c_n] \otimes E[a_1, a_3, \ldots, a_{2n-1}] \Longrightarrow \mathbb{Z},$$

with $c_i \in E_2^{2i,0}$ and $a_{2j-1} \in E_2^{0,2j-1}$. It is known that for $i < n$ we have

$$E_{2i+1}^{**} = E_{2i+2}^{**} = \mathbb{Z}[c_{i+1}, \ldots, c_n] \otimes E[a_{2i+1}, \ldots, a_{2n-1}]$$

with $d_{2i+1} = 0$ and $d_{2i+2}(a_{2i+1}) = c_{i+1}$. This leaves $E_{2n} = E_\infty = \mathbb{Z}$.

Example 3.7. Let G be a finite group of order 2^{2n+1} where the centre Z has order 2 and the quotient $V = G/Z$ is elementary abelian. (These are called *extraspecial* 2-groups.) We have a fibration

$$BZ \to BG \to BV.$$

As Z is central, the group $V = \pi_1(BV)$ acts trivially on BZ and we have a Serre spectral sequence in mod 2 cohomology

$$H^*(BV) \otimes H^*(BZ) \Longrightarrow H^*(BG).$$

Here $BZ = \mathbb{R}P^\infty$ and so $H^*(BZ) = \mathbb{F}[z]$ (where $\mathbb{F}$ means $\mathbb{Z}/2$ and $|z| = 1$). If we choose an isomorphism $V = C_2^{2d}$ we also get $H^*(BV) = \mathbb{F}[x_1, \ldots, x_{2d}]$. More invariantly, we can say that $H^*(BV)$ is the symmetric algebra $\mathbb{F}[V^*]$ generated by $H^1(BV) = V^* = \mathrm{Hom}(V, \mathbb{F})$. There is a quadratic form $q_0 \colon V \to \mathbb{F}$ given by $g^2 = z^{q_0(v)}$ for any $g \in G$ that maps to $v \in V$. This can be regarded as an element of $(V^* \otimes V_*)_{\Sigma_2} = H^2(BV)$, and it works out that $d_2(z) = q_0$, leaving

$$E_3 = \mathbb{F}[z^2] \otimes \mathbb{F}[V^*]/q_0.$$

Now define $q_i \in H^{2^i+1}(BV)$ recursively by $q_{i+1} = Sq^{2^i} q_i$, and $J_i = (q_j \mid j < i)$. It can be shown that for $i < d$ we have

$$E_{2^{i-1}+2} = \cdots = E_{2^i+1} = \mathbb{F}[z^{2^i}] \otimes \mathbb{F}[V^*]/J_i$$

with $d_{2^i+1}(z^{2^i}) = q_i$. For $i \geq d$ it turns out that $q_i \in J_d$ and there are no further differentials. We end up with $E_\infty = \mathbb{F}[z^{2^d}] \otimes \mathbb{F}[V^*]/J_d$.

For this to be compatible with the exactness properties of a spectral sequence, it must be true that q_i is not a zero divisor in $\mathbb{F}_p[V^*]/J_i$, for $i = 0, \ldots, d-1$. This can be proved directly, and was part of Quillen's proof that the spectral sequence works as described.

4. The Eilenberg-Moore spectral sequence

Suppose we have a homotopy pullback square

$$\begin{array}{ccc} W & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Z \end{array}$$

This makes H^*X and H^*Y into algebras (and thus modules) over H^*Z, so we can define groups $\mathrm{Tor}_{pq}^{H^*Z}(H^*X, H^*Y)$. There is then a spectral sequence

$$\mathrm{Tor}_{st}^{H^*Z}(H^*X, H^*Y) \Longrightarrow H^{t-s}W.$$

Example 4.1. Consider the square

$$\begin{array}{ccc} S^{2n+1} & \longrightarrow & 1 \\ \downarrow & & \downarrow \\ \mathbb{C}P^n & \longrightarrow & \mathbb{C}P^\infty. \end{array}$$

This gives an Eilenberg-Moore spectral sequence

$$\mathrm{Tor}_{st}^{\mathbb{Z}[x]}(\mathbb{Z}, \mathbb{Z}[x]/x^{n+1}) \Longrightarrow E[u],$$

where $|x| = 2$ and $|u| = 2n + 1$. We have a projective resolution

$$(P_1 \to P_0 \to \mathbb{Z}[x]/x^{n+1}) = (\mathbb{Z}[x]a \xrightarrow{d} \mathbb{Z}[x] \xrightarrow{\epsilon} \mathbb{Z}[x]/x^{n+1})$$

where $d(a) = x^{n+1}$. Here a has internal degree $2n+2$ (so that d preserves degrees) and cohomological degree 1 (because it lies in P_1). The relevant Tor groups are the homology groups of the complex $\mathbb{Z} \otimes_{\mathbb{Z}[x]} P_* = (\mathbb{Z}a \xrightarrow{0} \mathbb{Z})$, or in other words $\mathbb{Z}\{1, a\}$, with $1 \in \mathrm{Tor}_{00}$ and $a \in \mathrm{Tor}_{1,2n+2}$. There is no room for differentials, and a represents an element in $H^{2n+1}S^{2n+1}$ as expected.

Example 4.2. Consider the square

$$\begin{array}{ccc} U(n) & \longrightarrow & 1 \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & BU(n). \end{array}$$

We have $H^*BU(n) = \mathbb{Z}[c_1, \dots, c_n]$ (with $|c_i| = 2i$) and $H^*U(n) = E[a_1, a_3, \dots, a_{2n-1}]$ (with $|a_i| = i$). This gives a spectral sequence

$$\mathrm{Tor}_{**}^{\mathbb{Z}[c_1,\dots,c_n]}(\mathbb{Z}, \mathbb{Z}) \to E[a_1, \dots, a_{2n-1}].$$

To see how this works, consider the ring

$$R_{**} = \mathbb{Z}[c_1, \dots, c_n] \otimes E[b_1, \dots, b_n].$$

We give this the bigrading $|c_i| = (0, 2i)$ and $|b_i| = (1, 2i)$. We then define $d\colon R_{st} \to R_{s-1,t}$ by $d(c_i) = 0$ and $d(b_i) = c_i$ and the Leibniz rule $d(xy) = (dx)y + (-1)^s x\, dy$ for $x \in R_{st}$. It is clear that R is free as a module over $H^*BU(n)$. We also claim that $H_*(R_*; d) = \ker(d)/\operatorname{image}(d) = \mathbb{Z}$. This can be seen directly when $n = 1$. The general case is essentially the tensor product of n copies of the $n = 1$ case, so the claim follows by the Künneth theorem. This means that R_{**} gives a projective resolution of $\mathbb{Z}$ over $H^*BU(n)$, which we can use to calculate Tor. We see that $\mathbb{Z} \otimes_{H^*BU(n)} R_{**} = E[b_1, \dots, b_n]$, with trivial differential, so $\mathrm{Tor}^{H^*BU(n)}(\mathbb{Z}, \mathbb{Z}) = E[b_1, \dots, b_n]$. Here b_i has bidegree $(1, 2i)$ and so represents a class in $H^{2i-1}U(n)$, as expected.

5. The Rothenberg-Steenrod spectral sequence

Let G be a topological group, and suppose that G acts continuously on a space X. We can then form the Borel construction $X_{hG} = EG \times_G X$ (which is the same as X/G if the action is free). Note that the group structure makes H_*G a ring (possibly noncommutative), and $H_*(X)$ is a module over $H_*(G)$. There is then a Rothenberg-Steenrod spectral sequence

$$\mathrm{Tor}_{st}^{H_*G}(\mathbb{Z}, H_*(X)) \Longrightarrow H_{s+t}(X_{hG}).$$

In particular, by taking X to be a point (so $X_{hG} = BG$) we get a spectral sequence

$$\mathrm{Tor}_{st}^{H_*G}(\mathbb{Z}, \mathbb{Z}) \Longrightarrow H_{s+t}(BG).$$

Although ΩY is not actually a topological group, analogues of the above still work. In particular, if Y is connected then $B\Omega Y = Y$ and we get a spectral sequence

$$\mathrm{Tor}^{H_*\Omega Y}(\mathbb{Z},\mathbb{Z}) \Longrightarrow H_*Y.$$

Example 5.1. Consider the case $Y = S^{2n+1}$. Then $H_*\Omega Y$ is dual to $H^*\Omega Y$ and thus has a $\mathbb{Z}$ in each dimension $2nk$ and zeros elsewhere. It can be shown that $H_*\Omega Y$ is actually a polynomial algebra $\mathbb{Z}[w]$ with $|w| = 2n$. It follows that $\mathrm{Tor}^{H_*\Omega Y}(\mathbb{Z},\mathbb{Z}) = \mathbb{Z}\{1,u\}$ where u is in $\mathrm{Tor}_{1,2n}$ and thus represents a class in $H_{2n+1}S^{2n+1}$, as expected. There is no room for any differentials.

Example 5.2. Now instead take $Y = U(n)$. There is a map $\rho\colon \Sigma\mathbb{C}P^{n-1}_+ \to U(n)$, taking (z,L) to the map $g = z.1_L + 1_{L^\perp}$, with eigenvalue z on L and 1 on $L^\perp$. The homology $H_*U(n)$ is the exterior algebra generated by $\rho_*\widetilde{H}_*\Sigma\mathbb{C}P^{n-1}_+$, so $H_*U(n) = E[u_0,\dots,u_{n-1}]$ with $|u_i| = 2i+1$. Adjointly, we have a map $\rho^\#\colon \mathbb{C}P^{n-1} \to \Omega U(n)$, which gives a map from the symmetric algebra on $H_*\mathbb{C}P^n$ to $H_*\Omega U(n)$. This symmetric algebra is $\mathbb{Z}[b_0,\dots,b_{n-1}]$ (with $|b_i| = 2i$), and it is known that in fact $H_*\Omega U(n) = \mathbb{Z}[b_0,\dots,b_{n-1}][b_0^{-1}]$. The module $\mathbb{Z} = H_*(\text{point})$ is the quotient of $H_*\Omega U(n)$ by $(b_0-1,b_1,\dots,b_{n-1})$. One can write down an explicit resolution as for polynomial algebras and check that $\mathrm{Tor}^{H_*\Omega U(n)}_{**}(\mathbb{Z},\mathbb{Z})$ is an exterior algebra on n generators. It follows that there can be no differentials in the Rothenberg-Steenrod spectral sequence.

6. The homotopy fixed point spectral sequence

Let $C = \langle c \mid c^2 = 1\rangle$ act on KU by complex conjugation. It is supposed to work out that $KU^{hC} = KO$. This gives a homotopy fixed point spectral sequence

$$E_2^{st} = H^s(C; KU^t) \Longrightarrow KO^{t+s} \qquad\qquad d_r\colon E_r^{st} \to E_r^{s+r,t-r+1}$$

Here $KU^* = \mathbb{Z}[\nu^{\pm1}]$ with $\nu \in KU^{-2}$ and $c(\nu) = -\nu$. Let x be the generator of $E_2^{2,0} = H^2(C;\mathbb{Z}) = \mathbb{Z}/2$ and let w be the generator of $E_2^{1,-2} = H^1(C;\mathbb{Z}\nu) = \mathbb{Z}/2$. We then have

$$E_2^{**} = \mathbb{Z}[\nu^{\pm2},x]/(2x) \oplus (\mathbb{Z}/2)[\nu^{\pm2},x]w.$$

I think it works out that $w^2 = \nu^2 x$, so we can rewrite this as

$$E_2^{**} = \mathbb{Z}[\nu^{\pm2},w]/(2w).$$

I think that $d_2 = 0$ and $d_3(\nu^2) = w^3$ but $d_3(w) = 0$ (so $d_3(x) = \nu^{-4}w^5 = x^2 w$). This gives

$$E_3^{**} = \mathbb{Z}[\nu^{\pm4}]\{1, 2\nu^2, w, w^2\}/(2w, 2w^2).$$

This leaves no room for further differentials (as $E_3^{s*} = 0$ for $* < 0$ or $* > 2$) so $E_3 = E_\infty$, and this agrees with KO^*. The element w represents the Hopf map $\eta \in \pi_1 KO$.

We can analyse kU^{hC} in the same way. Here we have

$$E_2^{**} = \mathbb{Z}[\nu^2,x]/(2x) \oplus (\mathbb{Z}/2)[\nu^2,x]w.$$

This is no longer generated by ν and w. We again have $d_3(\nu^2) = w^3$ and $d_3(x) = x^2 w$. The E_3 page is truncated to $t \le 0$, and the d_3-cycles x^{2i+2}, which used to be hit by $\nu^{-2}x^{2i}w$, are no longer hit. We end up with

$$\pi_*(kU^{hC}) = \pi_*(kO) \oplus \mathbb{F}_2\{x^{2i+2} \mid i \ge 0\}.$$

Here x^{2i+2} represents a class in $\pi_{-2i-2}(kU^{hC})$.

7. The Atiyah-Hirzebruch spectral sequence

For any space X and any generalised cohomology theory R^* there is an Atiyah-Hirzebruch spectral sequence

$$E_2^{st} = H^s(X; R^t) \Longrightarrow R^{s+t}X.$$

In the description of the E_2 term, R^t means $R^t(\text{point})$, which is π_{-t} of the representing spectrum. The differentials respect the R^*-module structure.

In particular, for any prime p (taken to be odd, for simplicity) and any $n > 0$ we have a theory $K(n)$ (called Morava K-theory) with $K(n)^* = \mathbb{F}_p[v_n^{\pm1}]$, where $v_n \in K(n)^{2-2p^n}$. This gives an Atiyah-Hirzebruch spectral sequence

$$H^*(X;\mathbb{F}_p) \otimes K(n)^* = H^*(X; K(n)^*) \Longrightarrow K(n)^*X.$$

It is known that $d_r = 0$ for $r < 2p^n - 2$ whereas $d_{2p^n-2}(x) = v_n Q_n(x)$ for $x \in H^*(X;\mathbb{F}_p)$. Here Q_n is the n'th Milnor Bockstein operation in the mod p Steenrod algebra, given inductively by $Q_0 = \beta$ and $Q_{i+1} = P^{p^i} Q_i - Q_i P^{p^i}$.

Example 7.1. Let X be such that $H^*(X;\mathbb{F}_p)$ is concentrated in even degrees. Then each page of the spectral sequence for $K(n)^*X$ is concentrated in (even,even) bidegree and thus in even total degree. The differentials d_r all shift total degree by one, so they are zero. It follows that $E_2 = E_\infty$ and that the associated graded for the natural filtration of $K(n)^*X$ is isomorphic to $H^*(X;\mathbb{F}_p) \otimes K(n)^*$. This applies to $X = \mathbb{C}P^\infty$, for example.

Example 7.2. It is known that the differentials in the AHSS are always torsion-valued. Thus, if the E_2-page is torsion-free, then the spectral sequence must collapse. This applies to the AHSS for $MU^*U(n)$, for example.

Example 7.3. Consider $X = BC_p$, where $H^*(BC_p;\mathbb{F}_p) = \mathbb{F}_p[x] \otimes E[a]$. We have $\beta(a) = x$ and $\beta(x^j) = 0$ and $P^i(x^j) = \begin{pmatrix} j + (p-1)i \\ j \end{pmatrix} x^{j+(p-1)i}$, and it follows that $Q_i(ax^j) = x^{j+p^i}$ and $Q_i(x^j) = 0$. Thus, in the spectral sequence for $K(n)^*BC_p$ we have

$$E_2 = \mathbb{F}_p[x, v_n^{\pm 1}] \otimes E[a],$$

with $x \in E_2^{20}$ and $a \in E_2^{10}$ and $v_n \in E_2^{0,2-2p^n}$. The first differential is given by $d_{2p^n-2}(a) = v_n x^{p^n}$ and $d_{2p^n-2}(x) = 0$, which leaves

$$E_{2p^n-1} = \mathbb{F}_p[x, v_n^{\pm 1}]/x^{p^n}.$$

This is concentrated in the vertical band $0 \leq s \leq 2p^n - 2$, and all remaining differentials are so long that they must either start or end outside this band. It follows that $E_\infty = E_{2p^n-1}$. It can be shown that there are no filtration issues and so $K(n)^*BC_p$ is isomorphic to E_∞.

Example 7.4. Consider instead the AHSS for KU^*BC_2. It is standard that $H^*BC_2 = H^*\mathbb{R}P^\infty = \mathbb{Z}[x]/2x$, with $|x| = 2$. Our E_2 term is just $H^*(BC_2;K^*) = \mathbb{Z}[\nu^{\pm 1}, x]/2x$, with $\nu \in E_2^{0,-2}$ and $x \in E_2^{2,0}$. This is concentrated in even total degree, so there are no differentials. However, there are strong filtration effects. To explain this, put $y = \nu x \in K^0BC_2$. It turns out that $KU^0BC_2 = \mathbb{Z} \oplus \mathbb{Z}_2 y$, where $\mathbb{Z}_2$ denotes the 2-adic integers and $y^2 = 2y$. The natural filtration of K^0BC_2 is given by $F^{2i-1} = F^{2i} = (y^i)$, which is the same as $(2^{i-1}y)$ for $i > 0$. The associated graded is thus $F^0/F^1 = \mathbb{Z}$ and $F^{2i}/F^{2i+1} = (\mathbb{Z}/2).y^i$, which gives the terms $E_\infty^{2i,-2i}$ in the spectral sequence. Multiplication by powers of ν gives everything else.

8. The Adams spectral sequence

Let $\mathcal{A}$ denote the mod 2 Steenrod algebra. If we let H denote mod 2 cohomology then $H^*(X)$ is naturally an $\mathcal{A}$-module for any spectrum X, and there is a classical Adams spectral sequence

$$\mathrm{Ext}_{\mathcal{A}}^{st}(H^*(X), \mathbb{F}) \Longrightarrow \pi_{t-s}(X)_2^\wedge.$$

In particular, we get a spectral sequence

$$\mathrm{Ext}_{\mathcal{A}}^{st}(\mathbb{F}, \mathbb{F}) \Longrightarrow \pi_{t-s}(S^0)_2^\wedge.$$

Here the groups $\pi_k(S)$ are finite for $k > 0$, so the 2-adic completion simply replaces $\pi_0(S^0) = \mathbb{Z}$ by $\mathbb{Z}_2$ and $\pi_k(S^0)$ by the 2-torsion part of $\pi_k(S^0)$ for $k > 0$.

9. Other good examples

- SSS, RSSS and EMSS for $\Omega U(n) \to 1 \to U(n)$.
- ASS for MO_* and MU_*. Small parts of ASS for $\pi_*(S^0)$.
- Weierstrass SS for $\pi_*(TMF)$.
- EHPSS

10. Constructions

Let A be an abelian group with a self-map $d\colon A \to A$ satisfying $d^2 = 0$. Suppose we have a filtration

$$A \geq F_{k-1} \geq F_k \geq F_{k+1} \geq \cdots$$

with $dF_k \leq F_k$, and suppose for simplicity that $F_k = A$ for $k \ll 0$ and $F_k = 0$ for $k \gg 0$. Put

$$\begin{aligned} W_k^r &= \{a \in F_k \mid da \in F_{k+r-1}\} \\ Z_k^r &= W_k^r + F_{k+1} \\ B_k^r &= (d(F_{k-r+2}) \cap F_k) + F_{k+1} = d(W_{k-r+2}^{r-1}) + F_{k+1} \\ E_k^r &= Z_k^r / B_k^r = (W_k^r + B_k^r)/B_k^r. \end{aligned}$$

Note that

$$\begin{aligned} Z_k^1 &= F_k \\ B_k^1 &= F_{k+1} \\ E_k^1 &= F_k / F_{k+1} \end{aligned}$$

and

$$F_{k+1} \leq B_k^r \leq B_k^{r+1} \leq (dA \cap F_k) + F_{k+1} \leq Z_k^{r+1} \leq Z_k^r \leq F^k.$$

Lemma 10.1. *There is a well-defined map $d^r\colon E_k^r \to E_{k+r-1}^r$ given by*

$$d^r(a + B_k^r) = da + B_{k+r-1}^r$$

for $a \in W_k^r$.

Proof. First, if $a \in W_k^r$ then $da \in F_{k+r-1}$ and $d^2(a) = 0$ so $da \in W_{k+r-1}^r$ so $da + B_{k+r-1}^r \in E_{k+r-1}^r$. Next, if we have another representation $a + B_k^r = a' + B_k^r$ with $a' \in W_k^r$, then we must have $a' = a + b$ for some $b \in W_k^r \cap B_k^r$. As $b \in B_k^r$ we have $b = du + v$, where $u \in F_{k-r+2}$ and $du \in F_k$ and $v \in F_{k+1}$. It follows that $da' - da = db = dv$, and $db \in F_{k+r-1}$ and $dv \in d(F_{k+1})$. From the definitions we have $B_{k+r-1}^r = d(F_{k+1}) + F_{k+r-1}$, so $da' - da \in B_{k+r-1}^r$ as required. ∎

Lemma 10.2. $\ker(d^r\colon E_k^r \to E_{k+r-1}^r) = Z_k^{r+1}/B_k^r = \{a + B_k^r \mid a \in W_k^{r+1}\}$.

Proof. Given an element $u = a + B_k^r$ with $a \in W_k^{r+1} \leq W_k^r$, we have $d^r u = da + B_{k+r-1}^r$, but $da \in F_{k+r} \leq B_{k+r-1}^r$ so $d^r u = 0$. Conversely, suppose we have an element $u \in E_k^r$ with $d^r u = 0$. We can represent u as $u = a + B_k^r$ for some $a \in W_k^r$, and then we must have

$$da \in B_{k+r-1}^r = (d(F_{k+1}) \cap F_{k+r-1}) + F_{k+r}.$$

We can thus choose $b \in F_{k+1}$ and $c \in F_{k+r}$ with $da = db + c$ and $db \in F_{k-r+1}$. It follows that $b \in W_k^r \cap B_k^r$, so the element $a' = a - b$ again lies in W_k^r and $u = a' + B_k^r$. We have $da' = c \in F_{k+r}$, so $a' \in W_k^{r+1}$, so $u \in Z_k^{r+1}/B_k^r$ as claimed. ∎

Lemma 10.3. *$img(d^r\colon E_{k-r+1}^r \to E_k^r) = B_k^{r+1}/B_k^r$.*

Proof. The relevant image is by definition $(d(W_{k-r+1}^r) + B_k^r)/B_k^r$. On the other hand, we have $B_k^{r+1} = d(W_{k-r+1}^r) + F_{k+1}$. The claim follows easily. ∎

Corollary 10.4. *$(d^r)^2 = 0$, and $E_*^{r+1} = H(E_*^r, d^r)$.*

References

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