

**Groups and Symmetry**

Answer **four** questions. You are advised **not** to answer more than four questions: if you do, only your best four will be counted.

**1** Let  $I_2$  be the group of isometries of  $\mathbb{R}^2$ , and let  $\psi: I_2 \rightarrow O_2$  be the usual homomorphism, so  $\psi(f) = A$  if  $f(x) = Ax + a$  for all  $x$ .

(i) Let  $f = G_{L,a} \in I_2$  be a glide-reflection. Show that  $f^2 = T_{2a}$  and that  $L = \{x \in \mathbb{R}^2 \mid f(x) = x + a\}$ . Show that  $\psi(f) = S_\theta$ , where  $\theta/2$  is the angle between the  $x$ -axis and  $L$ . **(9 marks)**

(ii) Put  $b = f(0)$  and  $g = T_b S_\theta$  and  $h = g^{-1}f$ . Prove that  $\psi(h) = 1$  and  $h(0) = 0$  and deduce that  $f = g$ . **(7 marks)**

(iii) Put  $f(x, y) = (2 + y, x)$ ; you may assume that this is a glide-reflection. Find  $a$  and  $L$  such that  $f = G_{L,a}$ . Find  $\theta$  and  $b$  such that  $f = T_b S_\theta$ . **(9 marks)**

**2** (i) Let  $G$  be a finite subgroup of  $SO_2$ . Prove that  $G = C_n$  for some  $n > 0$ . **(8 marks)**

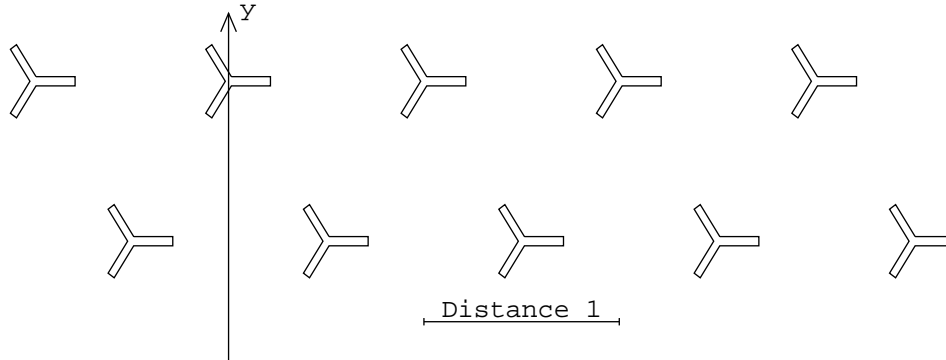
(ii) Suppose that  $a \in \mathbb{R}^2$  and  $\theta \in \mathbb{R}$ . Let  $f$  be the isometry  $R_\pi R_{a,\theta} R_\pi^{-1} R_{a,\theta}^{-1}$ .

(a) Show that  $f$  is a translation and that  $f(a) = 2R_\theta(a) - a$ . **(5 marks)**

(b) Suppose that  $0 < \theta < 2\pi$  and that there is a finite subgroup  $H$  of  $I_2$  such that  $R_\pi \in H$  and  $R_{a,\theta} \in H$ . Prove that  $f = 1$  and that  $a = 0$ . **(5 marks)**

(iii) Let  $K$  be a finite subgroup of  $SO_3$ . Define the set  $P$  of poles of  $K$ . Show that if  $|P| = 2$  then  $K$  is cyclic. **(7 marks)**

3 Let  $G$  be the isometry group of the infinite wallpaper pattern, a portion of which is illustrated below. (A copy of the diagram on white paper is provided; if you wish, you may write on it and hand it in with your answer.)



(i) Describe geometrically *all* the translations, rotations and reflections in  $G$ . State clearly the vectors of the translations, the centres and angles of the rotations, and the lines of the reflections. Specify one more element of  $G$  that is not a translation, rotation or reflection. Justify your answers. **(9 marks)**

(ii) Find a list of four isometries that generate  $G$ . Justify your answer. **(11 marks)**

(iii) State clearly the definition of the point group of the pattern. Find  $n$  and  $\theta$  such that the point group is equal to  $R_\theta D_n R_\theta^{-1}$ . Justify your answer. **(5 marks)**

- 4 (i) State the Sylow theorems. *(5 marks)*
- (ii) Let  $G$  be a group of order 33.
- (a) Show that  $G$  has a normal subgroup  $N$  of order 11. *(5 marks)*
- (b) What are the orders of  $\text{Aut}(N)$  and  $G/N$ ? Show that if  $P$  is a Sylow 3-subgroup of  $G$  then every element of  $P$  commutes with every element of  $N$ . *(7 marks)*
- (c) Deduce that  $G \simeq C_{11} \times C_3$ . *(8 marks)*
- 5 Let  $C$  be a cube centred at the origin in  $\mathbb{R}^3$ , and write  $G = \text{Dir}(C)$ .
- (i) Describe a set of four things on which  $G$  acts, and explain why this gives a homomorphism  $\phi: G \rightarrow S_4$ . *(5 marks)*
- (ii) Prove that  $\phi$  is injective. *(9 marks)*
- (iii) Let  $x$  be a point on the surface of  $C$  that lies on an edge, close to a corner but not at the corner. Prove that the orbit  $Gx$  has 24 elements. By considering the orders of various sets, deduce that  $\phi$  is surjective. *(5 marks)*
- (iv) Give a geometric description of elements  $g, h \in G$  such that  $\phi(g) = (1\ 2)$  and  $\phi(h) = (1\ 2\ 3)$ . *(6 marks)*

**End of Question Paper**