UNIVERSITY OF SHEFFIELD **PMA334**

PURE MATHEMATICS

Autumn Semester 1999–2000 $2\frac{1}{2}$ **hours**

Groups and Symmetry

*Answer **four** questions. You are advised **not** to answer more than four questions: if you do, only your best four will be counted.*

1 Let I_2 be the group of isometries of $\mathbb{R}^2$, and let $\psi\colon I_2 \longrightarrow O_2$ be the usual homomorphism, so $\psi(f) = A$ if $f(x) = Ax + a$ for all x.

(i) Let $f = G_{L,a} \in I_2$ be a glide-reflection. Show that $f^2 = T_{2a}$ and that $L = \{x \in \mathbb{R}^2 \mid f(x) = x + a\}$. Show that $\psi(f) = S_\theta$, where $\theta/2$ is the angle between the x-axis and L. ***(9 marks)***

(ii) Put $b = f(0)$ and $g = T_b S_\theta$ and $h = g^{-1} f$. Prove that $\psi(h) = 1$ and $h(0) = 0$ and deduce that $f = g$. ***(7 marks)***

(iii) Put $f(x, y) = (2 + y, x)$; you may assume that this is a glide-reflection. Find a and L such that $f = G_{L,a}$. Find θ and b such that $f = T_b S_\theta$. ***(9 marks)***

2 (i) Let G be a finite subgroup of SO_2. Prove that $G = C_n$ for some $n > 0$. ***(8 marks)***

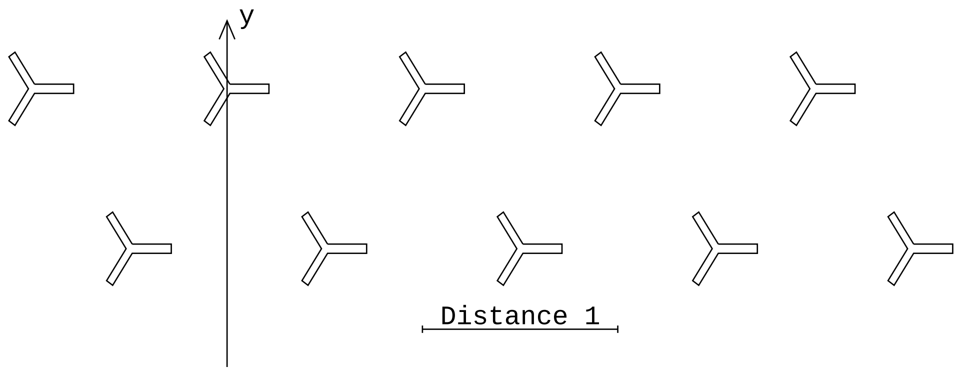
(ii) Suppose that $a \in \mathbb{R}^2$ and $\theta \in \mathbb{R}$. Let f be the isometry $R_\pi R_{a,\theta} R_\pi^{-1} R_{a,\theta}^{-1}$.

(a) Show that f is a translation and that $f(a) = 2R_\theta(a) - a$. ***(5 marks)***

(b) Suppose that $0 < \theta < 2\pi$ and that there is a finite subgroup H of I_2 such that $R_\pi \in H$ and $R_{a,\theta} \in H$. Prove that $f = 1$ and that $a = 0$. ***(5 marks)***

(iii) Let K be a finite subgroup of SO_3. Define the set P of poles of K. Show that if $|P| = 2$ then K is cyclic. ***(7 marks)***

3 Let G be the isometry group of the infinite wallpaper pattern, a portion of which is illustrated below. (A copy of the diagram on white paper is provided; if you wish, you may write on it and hand it in with your answer.)



(i) Describe geometrically *all* the translations, rotations and reflections in G. State clearly the vectors of the translations, the centres and angles of the rotations, and the lines of the reflections. Specify one more element of G that is not a translation, rotation or reflection. Justify your answers. ***(9 marks)***

(ii) Find a list of four isometries that generate G. Justify your answer. ***(11 marks)***

(iii) State clearly the definition of the point group of the pattern. Find n and θ such that the point group is equal to $R_\theta D_n R_\theta^{-1}$. Justify your answer. ***(5 marks)***

4 (i) State the Sylow theorems. ***(5 marks)***

(ii) Let G be a group of order 33.

(a) Show that G has a normal subgroup N of order 11. ***(5 marks)***

(b) What are the orders of $\text{Aut}(N)$ and G/N? Show that if P is a Sylow 3-subgroup of G then every element of P commutes with every element of N. ***(7 marks)***

(c) Deduce that $G \simeq C_{11} \times C_3$. ***(8 marks)***

5 Let C be a cube centred at the origin in $\mathbb{R}^3$, and write $G = \text{Dir}(C)$.

(i) Describe a set of four things on which G acts, and explain why this gives a homomorphism $\phi: G \longrightarrow S_4$. ***(5 marks)***

(ii) Prove that ϕ is injective. ***(9 marks)***

(iii) Let x be a point on the surface of C that lies on an edge, close to a corner but not at the corner. Prove that the orbit Gx has 24 elements. By considering the orders of various sets, deduce that ϕ is surjective. ***(5 marks)***

(iv) Give a geometric description of elements $g, h \in G$ such that $\phi(g) = (1\ 2)$ and $\phi(h) = (1\ 2\ 3)$. ***(6 marks)***

End of Question Paper