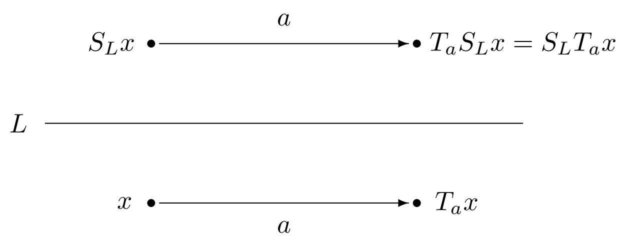
Groups and Symmetry — Exam solutions

(1) (i) We have $f = T_a S_L$, where a is parallel to L. It follows geometrically that $S_L(x+a) = S_L(x) + a$, or in other words that $S_L T_a = T_a S_L$



Thus $f^2 = T_a S_L T_a S_L = T_a T_a S_L S_L = T_{2a}$ (as $S_L^2 = 1$) .
It is also geometrically clear that the reflection of x across L is only equal to x when x lies on the line L. Thus

$$L = \{x \mid S_L(x) = x\} = \{x \mid S_L(x) + a = x + a\} = \{x \mid f(x) = x + a\}.$$

Finally, for any $b \in L$ we have $S_L = T_b S_\theta T_{-b}$, so

$$\psi(f) = \psi(T_{a+b} S_\theta T_{-b}) = \psi(T_{a+b})\psi(S_\theta)\psi(T_{-b}) = S_\theta.$$

(ii) We have $\psi(g) = \psi(T_b)\psi(S_\phi) = S_\phi = \psi(f)$, so $\psi(h) = \psi(g)^{-1}\psi(f) = 1$, which means that h is a translation . We also have $g(0) = b$ so $g^{-1}(b) = 0$ so $h(0) = g^{-1}f(0) = g^{-1}(b) = 0$. As h is a translation and $h(0) = 0$ we must have $h = 1$, so $f = g$.

(iii) Now take $f(x,y) = (2+y, x)$. We then have $f^2(x,y) = f(2+y,x) = (2+x, 2+y) = T_{(2,2)}(x,y)$. We thus have $2a = (2,2)$, so $a = (1,1)$. We also have

$$\begin{aligned} L &= \{(x,y) \mid f(x,y) = (x,y) + (1,1)\} \\ &= \{(x,y) \mid 2+y = x+1\ ,\ x = y+1\} \\ &= \{(x,y) \mid y = x-1\}. \end{aligned}$$

This line clearly has slope 1 and thus angle $\pi/4$ to the horizontal, so $\theta = \pi/2$. Finally, we have $b = f(0,0) = (2,0)$.

(2) (i) As G is finite, there are only finitely many angles ϕ in the range $(0, 2\pi]$ for which $R_\phi \in G$. Let θ be the least of these angles . If $\phi \geq 0$ and $R_\phi \in G$ we can choose a nonnegative integer k such that $k\theta \leq \phi < (k+1)\theta$ and then put $\psi = \phi - k\theta$. We have $0 \leq \psi < \theta \leq 2\pi$ and $R_\psi = R_\phi R_\theta^{-k} \in G$. As θ is the *smallest* angle in $(0, 2\pi]$ such that $R_\theta \in G$, we cannot have $\psi \in (0, 2\pi]$, so we must have $\psi = 0$. Thus $\phi = k\theta$. As $R_{2\pi} = 1$ is automatically in G, we can apply the above with $\phi = 2\pi$; we conclude that $2\pi = n\theta$ for some n, or equivalently $\theta = 2\pi/n$. We now see that $R_\phi \in G$ iff ϕ is a multiple of $2\pi/n$, so $G = C_n$.

(ii) We have $\psi(f) = R_\pi R_\theta R_\pi^{-1} R_\theta^{-1} = R_{\pi+\theta-\pi-\theta} = 1$, so f is a translation, say $f = T_b$. To determine b, we note that $R_\pi(x) = R_\pi^{-1}(x) = -x$

for all x and $R_{a,\theta}^{-1}(a) = a$, so

$$\begin{aligned} a + b &= f(a) \\ &= R_\pi R_{a,\theta} R_\pi^{-1} R_{a,\theta}^{-1}(a) \\ &= -R_{a,\theta}(-a) \\ &= -T_a R_\theta T_{-a}(-a) \\ &= -T_a R_\theta(-2a) \\ &= 2R_\theta(a) - a. \end{aligned}$$

It follows that $b = 2R_\theta(a) - 2a = 2(R_\theta - 1)(a)$.

As $f \in H$ and H is finite we see that f must have finite order, but nontrivial translations have infinite order so we must have $b = 0$. As R_θ is a nontrivial rotation in SO_2, it has no nonzero fixed points, so $(1 - R_\theta)$ is invertible . As $2(1 - R_\theta)(a) = 0$, we must have $a = 0$ as claimed.

(iii) We define

$$P = \{x \in \mathbb{R}^3 \mid \|x\| = 1 \text{ and } Ax = x \text{ for some } A \in G \setminus \{1\}\}.$$

Suppose that $|P| = 2$. Choose $x \in P$. Clearly $-x \in P$ also, and as $|P| = 2$ we must have $P = \{x, -x\}$. For any $A \in G \setminus \{1\}$ we know from Gauss's theorem that there exists $y \in \mathbb{R}^3$ with $\|y\| = 1$ and $Ay = y$. This means that $y \in P = \{x, -x\}$, so $x = y$ or $x = -y$. Either way we see that $Ax = x$; thus x is fixed under G . Now let U be the plane perpendicular to x. If $u \in U$ then $\langle Au, x\rangle = \langle Au, Ax\rangle = \langle u, x\rangle = 0$, so $Au \in U$ also . Thus, G is a finite subgroup of the group of rotations of the plane U, which is isomorphic to SO_2. We know from part (a) that a finite subgroup of SO_2 is cyclic, so G is cyclic .

(3) (i) Let u and v be the vectors indicated on the diagram, so $u = (1, 0)$ and $v = (1/2, \sqrt{3}/2)$. Then the translations in G are the maps T_{nu+mv} for $n, m \in \mathbb{Z}$. The nontrivial rotations in G are the maps $R_{a,\pm 2\pi/3}$, where a either lies in the centre of one of the motifs (such as the point marked O) or in the centre of the triangle formed by three adjacent motifs (such as the point marked P) . The lines of reflection are marked solidly on the diagram; for each motif, there are three such lines, one passing through each arm of the motif . The group G also contains the glide $G_{M,w}$ where M is the dotted line on the diagram (with equation $x = 1/4$) and $w = (0, \sqrt{3}/2)$.

(ii) Let L be as marked on the diagram, and put

$$\begin{aligned} R &= R_{2\pi/3} \\ S &= S_L \\ H &= \langle T_u, T_v, R, S\rangle. \end{aligned}$$

Clearly $T_u, T_v, R, S \in G$ so $H \leq G$. Now suppose that $f_0 \in G$. If $\delta(f) = 1$ we put $\epsilon = 0$, otherwise we put $\epsilon = 1$; we then put $f_1 = S^\epsilon f_0$. We have $f_1 \in G$ and $\delta(f) = 1$. Next, the map f_1 must carry the centre of the motif at O to the centre of some other motif, so $f(0) = nu + mv$ for some $n, m \in \mathbb{Z}$. Put $f_2 = T_u^{-n} T_v^{-m} f_1$, so

$f_2 \in G$ and $\delta(f) = 1$ and $f_2(0) = 0$. This means that f_2 is a rotation about 0, say $f_2 = R_\theta$. As f_2 carries the central motif to itself, it is clear that θ must be a multiple of $2\pi/3$, so $R_\theta = R^k$ for some k. Thus $R^k = T_u^{-n} T_v^{-m} S^\epsilon f_0$, so $f_0 = S^\epsilon T_v^m T_u^n R^k$, so $f_0 \in H$. Thus $H = G$ as claimed.

(iii) Any map $f \in I_2$ has the form $f(x) = Ax + b$ for some $A \in O_2$ and $b \in \mathbb{R}^2$. There is a homomorphism $\psi\colon I_2 \to O_2$ defined by $\psi(f) = A$. The point group of our pattern is the image of G under ψ. As G is generated by T, R and S, we see that $\psi(G)$ is generated by $\psi(T) = 1$, $\psi(R) = R_{2\pi/3}$ and $\psi(S) = S_0$. It follows that $\psi(G) = D_3 = R_0 D_3 R_0^{-1}$.

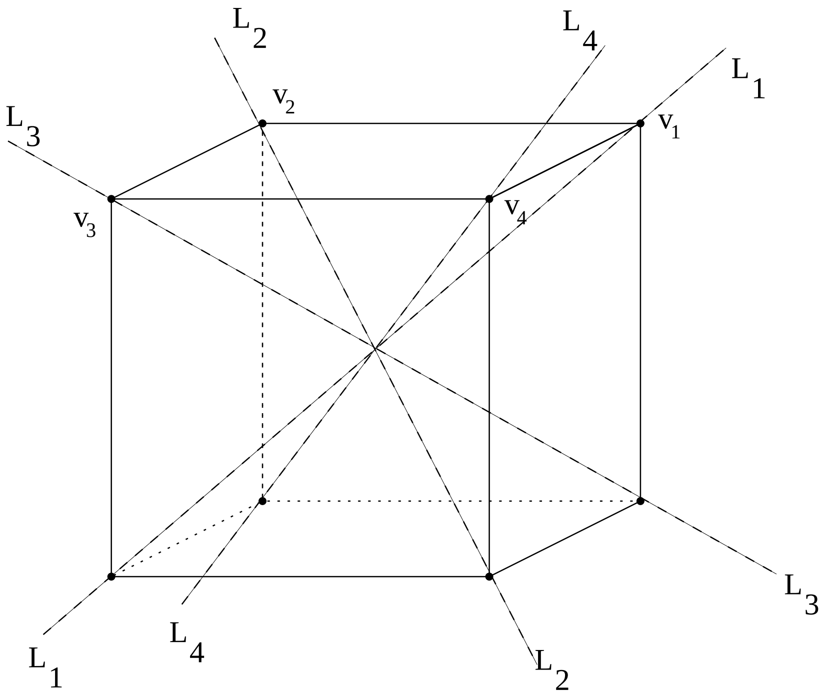
(4) (i) Let p be a prime, and let G be a finite group of order $p^v m$, where p does not divide m. Let n_p be the number of Sylow p-subgroups of G. Then n_p divides m and $n_p = 1 \pmod{p}$. Moreover, all Sylow p-subgroups are conjugate and every p-subgroup is contained in a Sylow p-subgroup.

(ii) (a) The number of Sylow 11-subgroups (that is, subgroups of order 11) is congruent to 1 mod 11 and divides $|G|/11 = 3$. As 1 is the only divisor of 3 that is congruent to 1 mod 11, there must be precisely one subgroup of order 11; call it N. If $g \in G$ then gNg^{-1} is again a subgroup of order 11, so it must be equal to N. Thus N is normal.

(b) As the order of N is prime, it must be a cyclic group, so $N \simeq C_{11}$. It follows that $\mathrm{Aut}(N) \simeq U(\mathbb{Z}/11)$, which is a group of order 10. On the other hand, we have $|G/N| = |G|/|N| = 33/11 = 3$. Now let P be a Sylow 3-subgroup of G, so $|P| = 3$ and $P \simeq C_3$. The action of P on N by conjugation gives a homomorphism $P \to \mathrm{Aut}(N)$. As the numbers $|P| = 3$ and $|\mathrm{Aut}(N)| = 10$ are coprime, this homomorphism must be trivial. As P acts trivially on N by conjugation, we see that P commutes with N.

(c) Let H be the subgroup of G generated by P and N. As H contains P we see that $|H|$ is divisible by 3. Similarly, it is divisible by 11, so it is divisible by $33 = |G|$, so we must have $G = H$. As P and N are Abelian and commute with each other and generate G, we see that G is Abelian. It follows that $G = PN$, and $P \cap N = 1$ (as P and N have coprime orders) so $G = P \times N$ as groups. Thus $G \simeq C_3 \times C_{11}$.

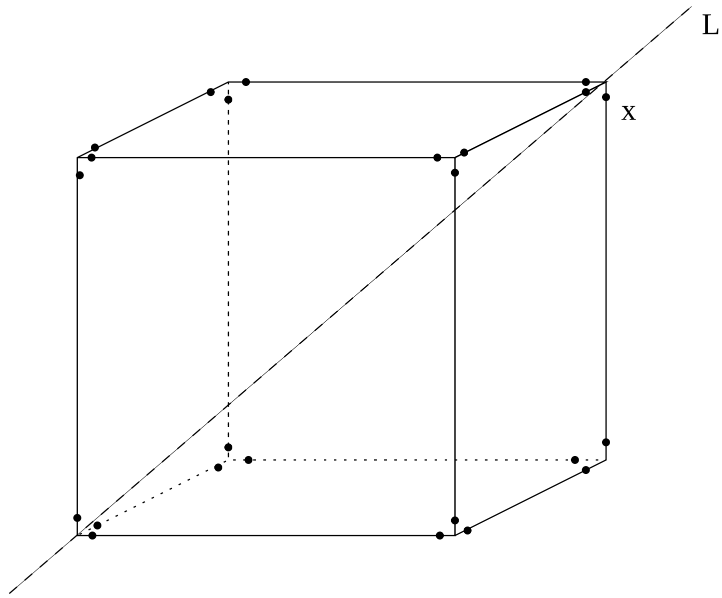
(5) (i) Let X be the set of long diagonals of C, labelled $L_1, \ldots, L_4$ as shown .



Fix $g \in G = \mathrm{Dir}(C)$. For each k, the line $g(L_k)$ is equal to one of the L_j's, say $g(L_k) = L_{\sigma(k)}$. This describes a permutation σ of $\{1, 2, 3, 4\}$, and we define $\phi\colon G \to S_4$ by $\phi(g) = \sigma$.

(ii) Suppose that $\phi(g) = 1$, so $g(L_k) = L_k$ for all k. Let the vectors $v_1, \ldots, v_4$ be as shown on the diagram, so v_k lies on L_k and has positive vertical component. As $g(L_k) = L_k$ we must have $g(v_k) = \pm v_k$, say $g(v_k) = \epsilon_k v_k$ with $\epsilon_k \in \{1, -1\}$. Suppose that at least three of the numbers ϵ_k are equal to -1. As any three of the v's form a basis, this implies that $g(x) = -x$ for all x, but this gives a contradiction, because the matrix -1 does not lie in SO_3. Thus at most two of the numbers ϵ_k is -1, so at least two of them are $+1$, so there is at least a two-dimensional space of vectors x for which $g(x) = x$. However, a nontrivial rotation of $\mathbb{R}^3$ has only a single line of fixed points, so g must be the identity element. This proves that ϕ is injective.

(iii) The orbit is as shown in the diagram. Indeed, by rotating around the long diagonal L, we get the two points close to x, then by rotating about the z-axis we get the remaining points near the top face, and then by rotating about the x-axis we get all the points near the bottom face.



There are three elements near each of the eight corners, so the orbit has size 24.

As the order of each orbit divides the order of the group, we see that $|G|$ is divisible by 24. However, the map $\phi\colon G \to S_4$ is injective, we have $|\phi(G)| = |G|$, so $|\phi(G)|$ is divisible by 24. However, $\phi(G)$ is a subgroup of S_4, so $|G| \leq |S_4| = 24$. We must thus have $|G| = 24$ and $\phi(G) = S_4$, so ϕ is surjective.

(iv) Let M be the line through the origin that is perpendicular to L_3 and L_4, and let g be the rotation through π around L. Then $g(L_3) = L_3$ and $g(L_4) = L_4$ and $g(L_1) = L_2$, so $\phi(g) = (1\ 2)$. Let h be the rotation through $2\pi/3$ around L_4, rotating clockwise as seen looking from v_4 to the origin; then $\phi(h) = (1\ 2\ 3)$.