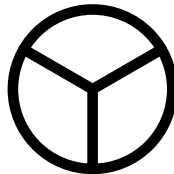


Groups and Symmetry - Sample Paper

Answer **four** questions. You are advised **not** to answer more than four questions: if you do, only your best four will be counted.

- 1
- (i) Define the group O_2 , and the elements R_θ and S_θ of O_2 .
 - (ii) Define the subgroups SO_2 , C_n and D_n of O_2 .
 - (iii) Let H be a finite subgroup of SO_2 . Prove that $H = C_n$ for some n .
 - (iv) Let K be a finite subgroup of O_2 , and suppose that K contains a reflection. Prove that $K = R_\theta D_n R_\theta^{-1}$ for some n and θ .
 - (v) Let X be the subset of \mathbb{R}^2 pictured below (with centre at the origin), and put $H = \text{Symm}(X)$. Find n and θ such that $H = R_\theta D_n R_\theta^{-1}$.



- 2
- (i) Define the group Isom_2 , and the function $\psi: \text{Isom}_2 \rightarrow O_2$.
 - (ii) Prove that ψ is a homomorphism.
 - (iii) Let H be a subgroup of Isom_2 . Define the groups $\text{Trans}(H) \leq \mathbb{R}^2$ and $\psi(H) \leq O_2$.
 - (iv) Explain what it means for H to be a *wallpaper group*.
 - (v) Let H be a wallpaper group, and suppose that $\psi(H) = C_n$. Show that $n \leq 6$. (You may assume that $\text{Trans}(H)$ has a shortest nonzero element, and also that $\|x - R_\theta(x)\| = 2 \sin(\theta/2) \|x\|$ for $0 \leq \theta \leq \pi$.)

3 (i) Let H be a subgroup of O_3 , and suppose that $-I \in H$. Prove that $H \simeq \{\pm 1\} \times G$ for some group $G \leq SO_3$.

(ii) Now let G be any finite subgroup of SO_3 .

(a) Define the set P of *poles* of G .

(b) Given a pole $v \in P$, define the *order* of v .

(iii) Put

$$X = \{(x, y, z) \in \mathbb{R}^3 \mid |x| \leq 1, |y| \leq 2, |z| \leq 3\}.$$

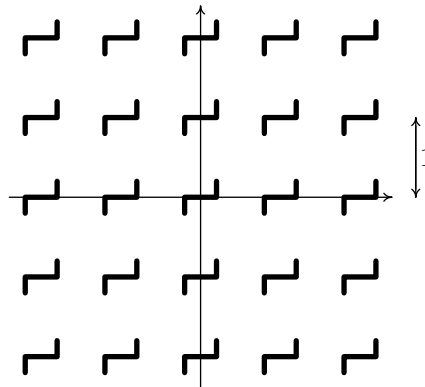
(a) Find three distinct, nontrivial rotations R_1, R_2 and R_3 in $\text{Dir}(X)$.

(b) Give formulae for $R_1(x, y, z), R_2(x, y, z)$ and $R_3(x, y, z)$.

(c) Using the previous part, show that the matrices R_i commute with each other, and that $R_1 R_2 R_3 = 1$.

(d) Show that $\text{Dir}(X) \simeq C_2 \times C_2$.

4 Let G be the isometry group of the infinite wallpaper pattern, a portion of which is illustrated below. (A copy of the diagram on white paper is provided; if you wish, you may write on it and hand it in with your answer.)



(i) Describe geometrically *all* the translations, reflections and rotations (if any) in G . State clearly the vectors of any translations, lines of any reflections, and the centres and angles of any rotations. **(9 marks)**

(ii) Find a list of three isometries that generate G . Justify your answer. **(11 marks)**

(iii) List all the vectors of minimal length in $\text{Trans}(G) \setminus \{0\}$. **(5 marks)**

- 5 (i) State the Sylow theorems. You should carefully define all the terms and notation used. *(5 marks)*
- (ii) Define the *center* of a group G .
- (iii) Let P be a group of order p^n , where p is prime and $n > 0$.
- (a) If P acts on a set X , show that $|\text{Fix}(X)| \equiv |X| \pmod{p}$.
- (b) Deduce that the center of P is nontrivial.
- (iv) Let G be a group of order $1225 = 5^2 \cdot 7^2$.
- (a) How many Sylow 5-subgroups does G have? Justify your answer.
- (b) How many Sylow 7-subgroups does G have? Justify your answer.
- (c) Prove that $G \simeq P \times Q$, where $|P| = 25$ and $|Q| = 49$.

End of Question Paper