UNIVERSITY OF SHEFFIELD **PMA334**

PURE MATHEMATICS

Autumn Semester 2003–2004 2 **hours** 30 **minutes**

Groups and Symmetry - Sample Paper

Answer ***four*** *questions. You are advised* ***not*** *to answer more than four questions: if you do, only your best four will be counted.*

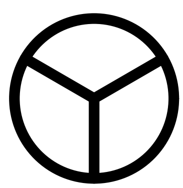
1 (i) Define the group O_2, and the elements R_θ and S_θ of O_2.

(ii) Define the subgroups SO_2, C_n and D_n of O_2.

(iii) Let H be a finite subgroup of SO_2. Prove that $H = C_n$ for some n.

(iv) Let K be a finite subgroup of O_2, and suppose that K contains a reflection. Prove that $K = R_\theta D_n R_\theta^{-1}$ for some n and θ.

(v) Let X be the subset of $\mathbb{R}^2$ pictured below (with centre at the origin), and put $H = \mathrm{Symm}(X)$. Find n and θ such that $H = R_\theta D_n R_\theta^{-1}$.



2 (i) Define the group Isom_2, and the function $\psi\colon \mathrm{Isom}_2 \to O_2$.

(ii) Prove that ψ is a homomorphism.

(iii) Let H be a subgroup of Isom_2. Define the groups $\mathrm{Trans}(H) \leqslant \mathbb{R}^2$ and $\psi(H) \leqslant O_2$.

(iv) Explain what it means for H to be a *wallpaper group*.

(v) Let H be a wallpaper group, and suppose that $\psi(H) = C_n$. Show that $n \leqslant 6$. (You may assume that $\mathrm{Trans}(H)$ has a shortest nonzero element, and also that $\|x - R_\theta(x)\| = 2\sin(\theta/2)\|x\|$ for $0 \leqslant \theta \leqslant \pi$.)

3 (i) Let H be a subgroup of O_3, and suppose that $-I \in H$. Prove that $H \simeq \{\pm 1\} \times G$ for some group $G \leqslant SO_3$.

(ii) Now let G be any finite subgroup of SO_3.

(a) Define the set P of *poles* of G.

(b) Given a pole $v \in P$, define the *order* of v.

(iii) Put

$$X = \{(x, y, z) \in \mathbb{R}^3 \mid |x| \leqslant 1\ ,\ |y| \leqslant 2\ ,\ |z| \leqslant 3\}.$$

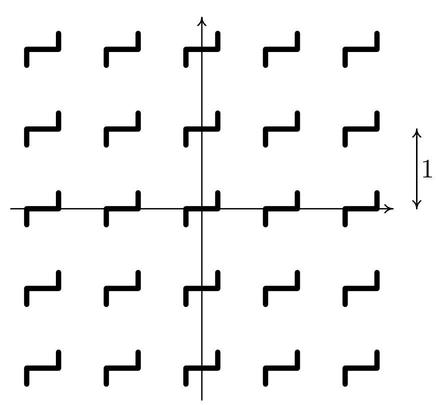
(a) Find three distinct, nontrivial rotations R_1, R_2 and R_3 in $\mathrm{Dir}(X)$.

(b) Give formulae for $R_1(x, y, z)$, $R_2(x, y, z)$ and $R_3(x, y, z)$.

(c) Using the previous part, show that the matrices R_i commute with each other, and that $R_1 R_2 R_3 = 1$.

(d) Show that $\mathrm{Dir}(X) \simeq C_2 \times C_2$.

4 Let G be the isometry group of the infinite wallpaper pattern, a portion of which is illustrated below. (A copy of the diagram on white paper is provided; if you wish, you may write on it and hand it in with your answer.)



(i) Describe geometrically *all* the translations, reflections and rotations (if any) in G. State clearly the vectors of any translations, lines of any reflections, and the centres and angles of any rotations. ***(9 marks)***

(ii) Find a list of three isometries that generate G. Justify your answer. ***(11 marks)***

(iii) List all the vectors of minimal length in $\mathrm{Trans}(G) \setminus \{0\}$. ***(5 marks)***

5 (i) State the Sylow theorems. You should carefully define all the terms and notation used. ***(5 marks)***

(ii) Define the *center* of a group G.

(iii) Let P be a group of order p^n, where p is prime and $n > 0$.

(a) If P acts on a set X, show that $|\mathrm{Fix}(X)| = |X| \pmod p$.

(b) Deduce that the center of P is nontrivial.

(iv) Let G be a group of order $1225 = 5^2.7^2$.

(a) How many Sylow 5-subgroups does G have? Justify your answer.

(b) How many Sylow 7-subgroups does G have? Justify your answer.

(c) Prove that $G \simeq P \times Q$, where $|P| = 25$ and $|Q| = 49$.

End of Question Paper