Groups and Symmetry Problem Set 1

Please hand in problems 1 and 5 on Tuesday 21st October.

Q1: Find $\mathrm{Symm}(X)$ and $\mathrm{Dir}(X)$ when $X \subseteq \mathbb{R}^2$ is

(a) The unit disc centred at the origin

(b) The isosceles triangle with vertices $(0, 2)$, $(-1, -3)$ and $(1, -3)$.

(c) The four points $(1, 1)$, $(1, -1)$, $(-1, 1)$ and $(-1, -1)$.

(d) The square with vertices $(-1, 2)$, $(-1, -2)$, $(3, 2)$ and $(3, -2)$.

Q2: Prove that D_n is generated by reflections.

(A group G is said to be *generated* by a subset Y if every $g \in G$ can be written in the form

$$g = y_1^{a_1} y_2^{a_2} \cdots y_n^{a_n}$$

for some $n \geq 1$, $y_1, y_2, \ldots, y_n \in Y$ and $a_1, a_2, \ldots, a_n \in \{\pm 1\}$.)

Q3: Recall that if H and K are groups the Cartesian product $H \times K$ is a group under the operation $(h, k)(h', k') = (hh', kk')$.

Show that if G is a group and $a, b \in G$ are distinct with the property that a, b and ab all have order 2 then $L = \{e, a, b, ab\}$ is a subgroup of G and L is isomorphic to $C_2 \times C_2$.

Show that if K is any group of order 4 then either $K \simeq C_4$ or $K \simeq C_2 \times C_2$, but not both. Which of these two alternatives hold for $K = D_2$?

Q4: Use the First Isomorphism Theorem to prove that $SO_2 \simeq \mathbb{R}/2\pi\mathbb{Z}$.

Q5: Put $H_n = \{A \in O_2 \mid A^n = 1\}$. Show that if n is odd then H_n is a finite subgroup of O_2 (which one?), but if n is even then H_n is not a subgroup at all.