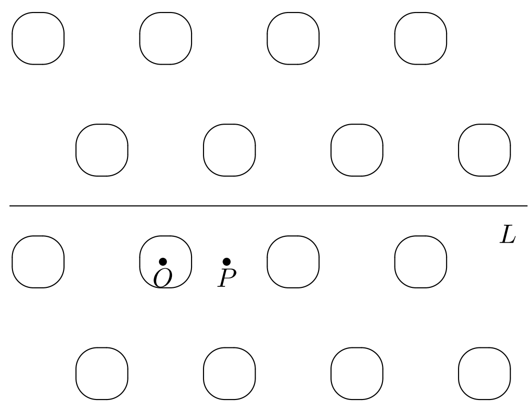
Groups and Symmetry Problem Set 3

Please hand in questions 1 and 4 on Tuesday November 4th.

Q1: Let X be the wallpaper pattern shown below. We have seen that $I(X) = \langle T_u, T_v, R_{\pi/3}, S_0 \rangle$, where $u = (1, 0)$ and $v = (1/2, \sqrt{3}/2)$. We also see geometrically that the rotation $R_{P,\pi}$ and the glide $G_{L,u/2}$ preserve X, so it must be possible to write $R_{P,\pi}$ and $G_{L,u/2}$ in terms of T_u, T_v, $R_{\pi/3}$ and S_0. Do this explicitly. [Hint: just follow through the steps in the proof that $I(X) = \langle T_u, T_v, R_{\pi/3}, S_0 \rangle$.]



Q2: Let H be a wallpaper group, and put $L = \{h(0) \mid h \in H\}$. Prove that $H \cap O_2 \leq \psi(H)$. Prove also that if $L = \text{Trans}(H)$, then $\psi(H) = H \cap O_2$.

Q3: Let H be a wallpaper group such that $\text{Trans}(H) = \{(2n, m) \mid n, m \in \mathbb{Z}\}$. Prove that $|\psi(H)| \leq 4$.

Q4:

(a) Let the group G act on the non empty set X. We say that G acts *transitively* on X if for all $x, y \in X$ there is an element $g \in G$ so that $g * x = y$.

Show that the following are equivalent

(1) G acts transitively on X

(2) For any $z \in X$ we have $G * z = X$

(3) For some $z \in X$ we have $G * z = X$.

(b) Decide which of the following actions are transitive.

(1) S_n acting naturally on $\{1, 2, \ldots, n\}$.

(2) D_4 acting on the square X_4.

(3) S_6 acting by conjugation on the set of elements of S_6 having order 3 (so $\theta * \phi = \theta\phi\theta^{-1}$).

(c) Let G be a group and H be a subgroup. Then G acts on $G/H = \{xH \mid x \in G\}$, the set of left cosets of H in G, by left multiplication: $g * xH = gxH$, for $g \in G$ and $xH \in G/H$. Show G acts transitively on G/H.

(d) If G acts on sets X_1 and X_2 by $\bullet\colon G \times X_1 \to X_1$ and $*\colon G \times X_2 \to X_2$ we say these actions are *equivalent* if there is a bijection $\phi\colon X_1 \to X_2$ such that $g * \phi(x) = \phi(g \bullet x)$ for all $g \in G$ and $x \in X_1$.

Show that if G acts transitively on a set X then this action is equivalent to one of G by left multiplication on G/H, for some subgroup H of G.