Groups and Symmetry Problem Set 4

Please hand in questions 1 and 4 on Tuesday 18th November.

Q1: Let $g\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the map $g(x, y, z) = (y, z, x)$.

(a) Prove that $g \in O_3$.
(b) Find a unit vector u with $g(u) = u$.
(c) Find the order of g and deduce that $g \in SO_3$.
(d) Show that g preserves the standard cube (with centre at the origin and edges of length 2 parallel to the x, y and z axes).
(e) Describe the effect of g geometrically.

Q2: In this problem we will find the coordinates of the vertices of a tetrahedron. We will place the tetrahedron with its centre at the origin and the first vertex v_1 on the z-axis. After choosing suitable units of length we may assume that $v_1 = (0, 0, \sqrt{3})$. We then rotate our coordinates if necessary so that the next vertex v_2 lies in the xz-plane, say $v_2 = (a, 0, -b)$. There are two more vertices; we call the one with positive y-coordinate v_3, and the one with negative y-coordinate v_4.

(a) Explain why the points v_3 and v_4 lie in the plane $z = -b$.
(b) Explain why $v_3 = (-a/2, a\sqrt{3}/2, -b)$, and give the corresponding formula for v_4.
(c) By considering the distances $d(O, v_i)$ and $d(v_i, v_j)$ show that $a^2 + b^2 = 3$ and $a^2 + (b + \sqrt{3})^2 = 3a^2$.
(d) Solve these equations for a and b, and obtain explicit expressions for the coordinates of v_2, v_3 and v_4.
(e) Consider the matrix

$$g = \begin{pmatrix} 1/3 & 0 & \sqrt{8}/3 \\ 0 & -1 & 0 \\ \sqrt{8}/3 & 0 & -1/3 \end{pmatrix}$$

Calculate $g(v_i)$ for $i = 1, 2, 3, 4$ and thus determine the vertex permutation induced by g.

Q3: We know that $\mathrm{Dir}(\mathrm{Tet}) \simeq A_4$ and that $\mathrm{Dir}(\mathrm{Cube}) \simeq S_4$. Find a geometric reason that Dir(Tet) is isomorphic to a subgroup of Dir(Cube). [Hint: can you see a tetrahedron inside the cube?]

Q4: Let M_1, M_2 and M_3 be the x, y and z-axes.

(a) Use these to define a homomorphism $\psi\colon \mathrm{Dir}(\mathrm{Cube}) \to S_3$.
(b) Describe some elements $g \in \mathrm{Dir}(\mathrm{Cube})$ and the corresponding permutations $\psi(g)$.
(c) Show that ψ is surjective.
(d) Show that the kernel of ψ is isomorphic to $C_2 \times C_2$.