Groups and Symmetry Problem Set 6

Q1: Let G be a finite group, and let X be a set with an action of G. Suppose that there is precisely one orbit, and that $|X| > 1$. Use the orbit counting theorem to show that there is an element $g \in G$ such that $\mathrm{Fix}(g) = \emptyset$.

Q2: Let G be a finite simple group (so there are no normal subgroups of G except for $\{1\}$ and G itself). Let p be a prime dividing the order of G, and let X be a set of order n on which the group acts. Suppose that the action is nontrivial, so there is and element $g \in G$ and an element $x \in X$ such that $gx \neq x$. Prove that $n \geq p$.

[**Hint:** Use X to define a homomorphism and consider its kernel. For which integers m does p divide $m!$?]

Q3: Show that every group of order 1225 is abelian.

Q4: The quaternion group of order 8 is the group

$$G = \{\pm 1, \pm i, \pm j, \pm k\}$$

with $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$, $i^2 = j^2 = k^2 = -1$. Show that if P and Q are any two nontrivial subgroups of G, then $P \cap Q$ is also nontrivial.

Q5: Let G be a group of order 18.

(a) Show that there is a normal subgroup $P \leq G$ with $|P| = 9$, and another subgroup $Q \leq G$ of the form $Q = \{1, h\}$ with $h^2 = 1$ (and so $h^{-1} = h$).

(b) Show that if $P \simeq C_9$ then $G \simeq C_2 \times C_9$ or $G \simeq D_9$.

(c) Now suppose that $P \simeq C_3 \times C_3$ (so in particular, P is abelian, and $x^3 = 1$ for all $x \in P$). For any $x \in P$, we put

$$\begin{aligned} x^h &= hxh = h^{-1}xh \\ x_+ &= x^2\,(x^h)^2 = xxhxhhxh = xxhxxh \\ x_- &= x^2\,x^h = xxhxh \\ P_+ &= \{x \in P \mid x^h = x\} \\ P_- &= \{x \in P \mid x^h = x^{-1}\}. \end{aligned}$$

(i) Show that $x_+ \in P_+$ and $x_- \in P_-$ and $x = x_+ x_-$.

(ii) Show that P_+ and P_- are subgroups of P, and that $P_+ \cap P_- = \{1\}$.

(iii) Deduce that $P \simeq P_+ \times P_-$.

(iv) Show that if $|P_+| = 9$ then $G \simeq C_3 \times C_3 \times C_2$.

(v) Show that if $|P_+| = 3$ then $G \simeq C_3 \times D_3$.

(The case $|P_+| = 1$ gives a new group, for which we do not yet have a name.)