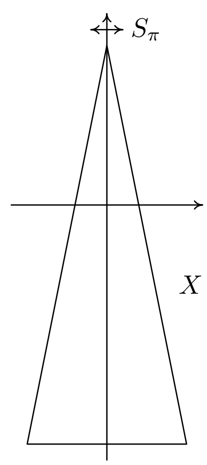
Groups and Symmetry Problem Set 1 — Solutions

Q1:

(a) Here $\mathrm{Symm}(X) = O_2$ and $\mathrm{Dir}(X) = SO_2$. This just means that the unit disc is invariant under any rotation about the origin, and under any reflection across a line through the origin, which is geometrically clear.

For a more algebraic proof, note that $X = \{x \in \mathbb{R}^2 \mid \|x\| \leq 1\}$. For any $A \in O_2$ and $x \in X$ we have $\|Ax\| = \|x\| \leq 1$ so $Ax \in X$; thus $AX \subseteq X$. Conversely, if $y \in X$ then $\|A^{-1}y\| = \|y\| \leq 1$ so the point $x := A^{-1}y$ lies in X. We have $y = Ax$ so $y \in AX$. This shows that $X \subseteq AX$, so $X = AX$, so $A \in \mathrm{Symm}(X)$. As A was an arbitrary element of O_2 we have $\mathrm{Symm}(X) = O_2$ and $\mathrm{Dir}(X) = \mathrm{Symm}(X) \cap SO_2 = O_2 \cap SO_2 = SO_2$, as claimed.

(b) Here it is evident that the only symmetry is under reflection across the y-axis, which lies at angle $\pi/2$ to the horizontal. Recall that S_θ is the reflection across the line at angle $\theta/2$ to the horizontal, so reflection across the y-axis is S_π. Thus $\mathrm{Symm}(X) = \{1, S_\pi\}$. This contains no rotations other than the identity, so $\mathrm{Dir}(X) = \{1\}$.

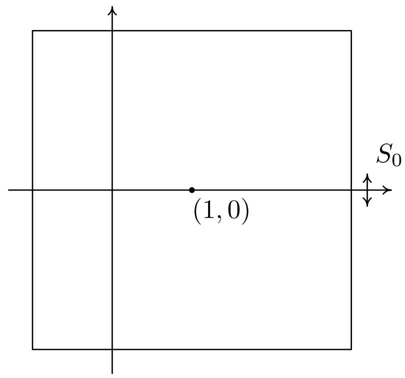


(c) Here X consists of the vertices of a square of side 2 with horizontal and vertical sides (note that this is different from our usual square X_4). We have $\mathrm{Symm}(X) = D_4$ and $\mathrm{Dir}(X) = C_4$. Indeed, it is clear that X is invariant under a rotation R_θ if and only if θ is a multiple of a quarter-turn, or in other words a multiple of $\pi/2 = 2\pi/4$. Thus if we put $R = R_{\pi/2}$ we find that

$$\mathrm{Dir}(X) = \{1 = R_0, R_{\pi/2}, R_\pi, R_{3\pi/2}\} = \{1, R, R^2, R^3\} = C_4.$$

It is also clear that X is unchanged if we reflect it across the x-axis, so $S_0 \in \mathrm{Symm}(X)$. If $A \in \mathrm{Symm}(X)$ is a reflection then AS_0 is a rotation and lies in $\mathrm{Symm}(X)$ so $AS_0 = R^k$ for some k so $A = AS_0S_0 = R^kS_0$ so $A \in D_4$. If $A \in \mathrm{Symm}(X)$ is a rotation we have seen that $A \in C_4 \subseteq D_4$, so again $A \in D_4$; thus $\mathrm{Symm}(X) \subseteq D_4$. As R and S_0 preserve X we also have $D_4 \subseteq \mathrm{Symm}(X)$, so $\mathrm{Symm}(X) = D_4$. Alternatively, we can just observe geometrically that there are four lines of reflectional symmetry at angles 0, $\pi/4$, $\pi/2$ and $3\pi/4$ to the x-axis, so the reflections in $\mathrm{Symm}(X)$ are S_0, $S_{\pi/2}$, S_π and $S_{3\pi/2}$.

(d) Here X is an off-centre square.



There is a lot of symmetry about the point $(1, 0)$ at the centre of the square. However, the question asks about $\mathrm{Symm}(X)$, which is by definition the group of symmetries about the point $(0, 0)$, and from that point of view the picture is much less symmetrical. In fact, the only symmetry is the reflection across the x-axis, so $\mathrm{Symm}(X) = \{1, S_0\}$ and $\mathrm{Dir}(X) = \{1\}$.

Q2: Write $R = R_{2\pi/n}$ and $S = S_0$ so

$$D_n = \{1, R, \ldots, R^{n-1}, S, RS, \ldots, R^{n-1}S\}.$$

Using the fact that $R_\alpha S_\beta = S_{\alpha+\beta}$ we see that $R^k S = R_{2k\pi/n} S_0 = S_{2k\pi/n}$, which is a reflection. Also, because $S^2 = 1$ we see that $R^k = (R^k S)S$. Here $R^k S$ and S are reflections lying in D_n, so R^k can be written as a product of two reflections lying in D_n. Thus every element in D_n is either a reflection or a product of reflections, so the reflections in D_n generate D_n as claimed.

Q3: We first claim that the elements e, a, b and ab are all distinct. Indeed, if $a = e$ or $b = e$ or $ab = e$ then a, b or ab would have order 1 rather than 2, contradicting our assumption. We also have $b \neq a$ by assumption. We cannot have $ab = a$, because if we did we could multiply on the left by a^{-1} to get $b = e$. Similarly, we cannot have $ab = b$, so all four elements are distinct, so L is a set of order 4.

The set L clearly contains e. As $a^2 = b^2 = (ab)^2 = e$ we have $a^{-1} = a$, $b^{-1} = b$, $(ab)^{-1} = ab$ and of course $e^{-1} = e$; thus L is closed under taking inverses. We also have $ab = (ab)^{-1} = b^{-1}a^{-1} = ba$ so a and b commute. Using this, it is easy to fill in the multiplication table as follows:

	e	a	b	ab
e	e	a	b	ab
a	a	e	ab	b
b	b	ab	e	a
ab	ab	b	a	e

(For example, $(ab)a = aba = aab = b$, which explains the entry in the row marked ab and the column marked a.) The table shows that L is closed under multiplication, so it is a subgroup. Recall that $C_2 = \{1, R\}$ where $R^2 = 1$. We can define $\phi\colon C_2 \times C_2 \to L$ by $\phi((R^i, R^j)) = a^i b^j$, so

$$\phi((1,1)) = e$$
$$\phi((R,1)) = a$$
$$\phi((1,R)) = b$$
$$\phi((R,R)) = ab$$

As $C_2 \times C_2 = \{(1,1), (R,1), (1,R), (R,R)\}$ we see that ϕ is a bijection.

As a and b commute we have

$$\begin{aligned}\phi((R^i,R^j))\phi((R^k,R^l)) &= a^ib^ja^kb^l \\ &= a^ia^kb^jb^l \\ &= a^{i+k}b^{j+l} \\ &= \phi((R^{i+k},R^{j+l})) \\ &= \phi((R^i,R^j)(R^k,R^l)),\end{aligned}$$

which shows that ϕ is a homomorphism and thus an isomorphism.

Now let K be a group of order 4. By Lagrange's theorem, every element $a \in K$ has order dividing 4 and thus equal to 1, 2 or 4. Only the identity element can have order 1 so the other 3 elements must have order 2 or 4. If there are no elements of order 4 then let a and b be any two distinct elements of order 2. Then ab is another element of K, which is not the identity because $a \neq b^{-1} = b$, so it must also have order 2. Using the first part of the question we deduce that $L = \{1, a, b, ab\}$ is a subgroup of K and is isomorphic to $C_2 \times C_2$. As $|L| = |K| = 4$ we must have $L = K$, so $K \simeq C_2 \times C_2$.

On the other hand, suppose that not all elements of $K \setminus \{1\}$ have order 2. Let a be an element of order 4, and put $L = \{1, a, a^2, a^3\}$. It is easy to see that L is a subgroup isomorphic to C_4, and $|L| = |K| = 4$ so $L = K$, so $K \simeq C_4$ as claimed.

We cannot have both $K \simeq C_2 \times C_2$ and $K \simeq C_4$, for in the first case all elements of K have order 2 whereas in the second case some elements have order 4.

Q4: Define $\phi\colon \mathbb{R} \to SO_2$ by $\phi(\alpha) = R_\alpha$. We have

$$\phi(\alpha)\phi(\beta) = R_\alpha R_\beta = R_{\alpha+\beta} = \phi(\alpha+\beta),$$

so ϕ is a homomorphism. Any element of SO_2 has the form $R_\alpha = \phi(\alpha)$ for some α, so ϕ is surjective. Thus, the First Isomorphism Theorem gives us an isomorphism $\overline{\phi}\colon \mathbb{R}/\ker(\phi) \to SO_2$. Moreover, we have $\alpha \in \ker(\phi)$ iff $\phi(\alpha) = 1$ iff $R_\alpha = R_0$ iff α is an integer multiple of 2π, so $\ker(\phi) = 2\pi\mathbb{Z}$. Thus $\mathbb{R}/2\pi\mathbb{Z} \simeq SO_2$ as claimed.

Q5: First suppose that n is odd, say $n = 2m+1$. For any θ we have $S_\theta^2 = 1$ so $S_\theta^n = (S_\theta^2)^m S_\theta = S_\theta \neq 1$, so $S_\theta \notin H_n$. On the other hand, we have $R_\theta^n = 1$ iff $n\theta = 2k\pi$ for some $k \in \mathbb{Z}$, iff $\theta = 2k\pi/n$ for some k, iff $R_\theta \in C_n$. Thus $H_n = C_n$, which is a finite subgroup of O_2.

Now suppose instead that n is even, say $n = 2m$. Then for all θ we have $S_\theta^n = (S_\theta^2)^m = 1$, so $S_\theta \in H_n$. For most θ we have $R_\theta^n \neq 1$, so $R_\theta \notin H_n$. Thus S_θ and S_0 lie in H_n but $S_\theta S_0 = R_\theta$ does not; this shows that H_n is not a subgroup.