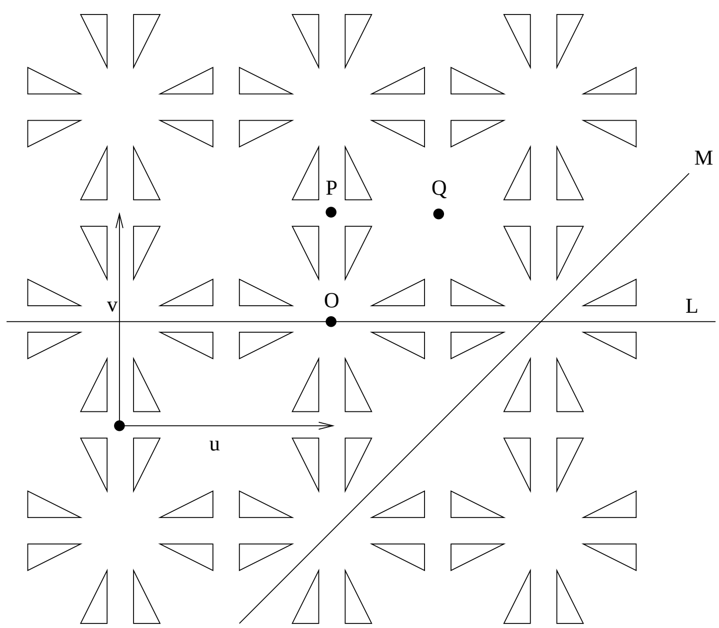
Groups and Symmetry Problem Set 2 — Solutions

Q1: First, we have $R_{a,\pi} = T_a R_\pi T_{-a}$, so $R_{a,\pi}(x) = a + R_\pi(x-a) = a - (x-a) = 2a - x$. If $y = 2a - x$ then $x = 2a - y$; this shows that $R_{a,\pi}^{-1}(y) = 2a - y$ and thus $R_{a,\pi}^{-1} = R_{a,\pi}$. Using this we have

$$\begin{aligned} R_{a,\pi}R_{b,\pi}R_{a,\pi}^{-1}R_{b,\pi}^{-1}(x) &= 2a - (2b - (2a - (2b - x))) \\ &= 2a - (2b - (2a - 2b + x)) \\ &= 2a - (4b - 2a - x)) \\ &= 4a - 4b + x, \end{aligned}$$

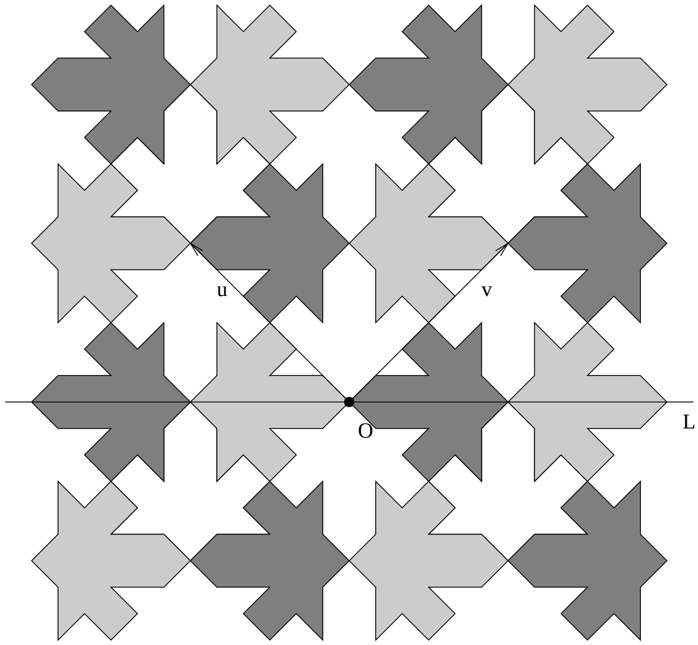
so $[R_{a,\pi}, R_{b,\pi}] = T_{4(a-b)}$. On the other hand, we saw in lectures that $[R_{a,\theta}, R_{b,\phi}] = T_{(1-R_\theta)(1-R_\phi)(a-b)}$. This is consistent because $1 - R_\pi$ is just twice the identity matrix, so $(1-R_\theta)(1-R_\phi)(a-b) = 4(a-b)$.

Q2: Pattern (a): Let R be rotation throught $\pi/2$ about O, and let S be reflection across L. It is clear that T_u, T_v, R and S preserve X, so $\langle T_u, T_v, R, S\rangle \leq I(X)$. Now suppose we have $f_0 \in I(X)$. If $\det(f_0) = -1$ we put $f_1 = Sf_0$, otherwise we put $f_1 = f_0$; either way we have $\det(f_1) = 1$ and $f_1 \in I(X)$. Clearly f_1 must send O to the centre of one of the motifs, so $f_1(O) = nu + mv + O$ for some $n, m \in \mathbb{Z}$. We put $f_2 = T_u^{-n}T_v^{-m}f_1$, so $f_2 \in I(X)$, $\det(f_2) = 1$ and $f_2(O) = O$. Thus f_2 is a rotation about O that preserves X; clearly the angle must be a multiple of $\pi/2$, so $f_2 = R^k$ for some k. We thus have $f_1 = T_v^m T_u^n R^k$ and either $f_0 = T_v^m T_u^n R^k$ or $f_0 = ST_v^m T_u^n R^k$. Thus $f_0 \in \langle T_u, T_v, R, S\rangle$, which proves that $I(X) = \langle T_u, T_v, R, S\rangle$.

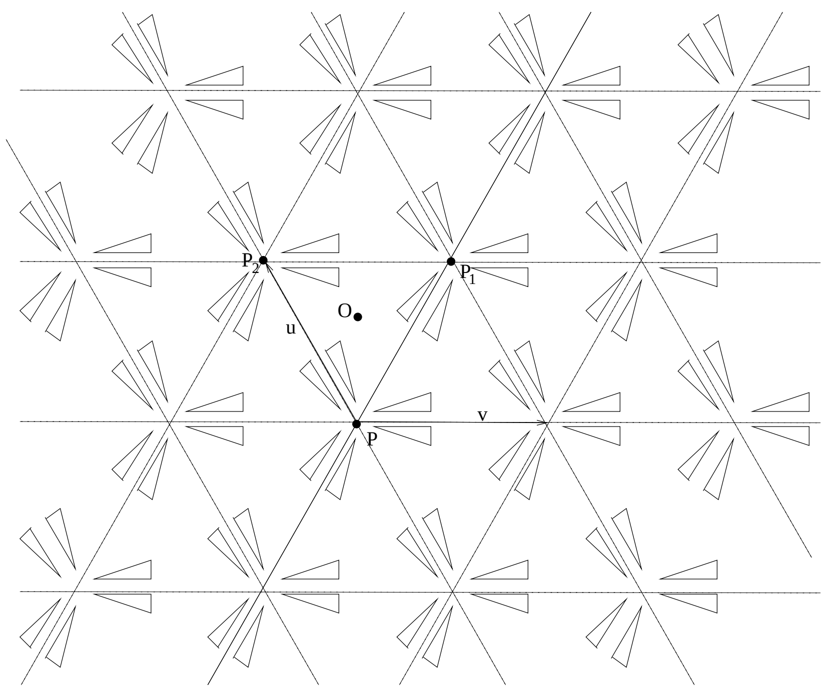


In particular, we see that S_M, $R_{P,\pi}$ and $R_{Q,\pi/2}$ all lie in $\langle T_u, T_v, R, S\rangle$. By following the above recipe we find that $S_M = ST_vT_uR$, $R_{P,\pi} = T_vR^2$ and $R_{Q,\pi/2} = T_uR$.

Pattern (b): Clearly $\langle T_u, T_v, S_L\rangle \leq I(X)$. Suppose that $f_0 \in I(X)$, and put $f_1 = S_Lf_0$ if $\det(f_0) = -1$ and $f_1 = f_0$ otherwise. Note that O is the point where the blunt ends of two white motifs meet, so $f_1(O)$ must also be the point of intersection of the blunt ends of two white motifs, so $f_1(O) = O + nu + mv$ for some $n, m \in \mathbb{Z}$. Put $f_2 = T_u^{-n}T_v^{-m}f_1$, so f_2 is a rotation around O that preserves the pattern X. There is only one dark grey motif adjacent to O, so f_2 must send that motif to itself, and this forces f_2 to be the identity. Thus either $f_0 = T_v^m T_u^n$ or $f_0 = S_L T_v^m T_u^n$, and in either case we have $f_0 \in \langle S_L, T_u, T_v \rangle$. This shows that $I(X) = \langle S_L, T_u, T_v \rangle$.



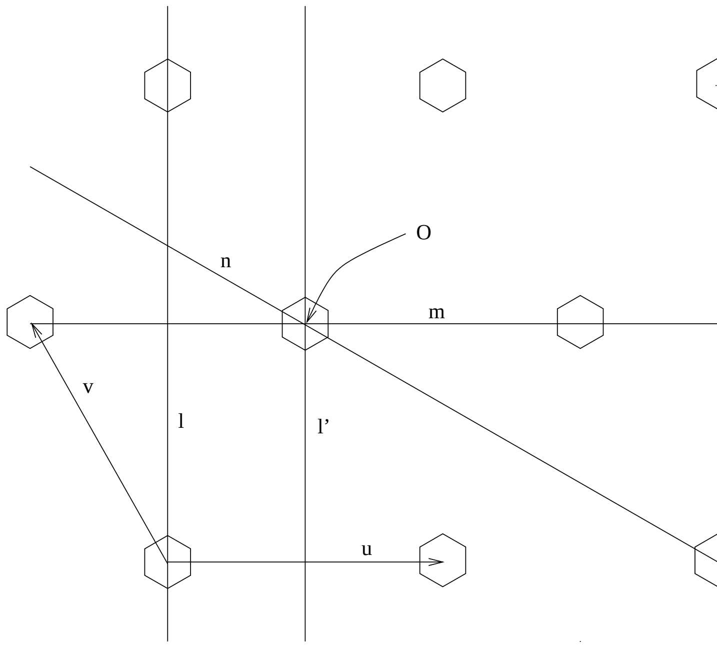
Pattern (c): Put $S = S_L$ and $R = R_{O,2\pi/3}$, so $\langle R, S \rangle \leq I(X)$.



As $\psi(S)$ is a reflection and $\psi(R)$ is a rotation we see that $\psi(RS) = \psi(R)\psi(S)$ is a reflection and thus that $\psi(RSRS) = \psi(RS)^2 = 1$, so $RSRS$ is a translation. As P lies on L we have $S(P) = P$ and $RS(P) = R(P) = P_1$. This lies on L again, so $SRS(P) = S(P_1) = P_1$, and it follows that $RSRS(P) = R(P_1) = P_2$. Thus $RSRS$ is a translation sending O to P_2, so we must have $RSRS = T_u$. Similarly, we have $SRSR = T_v$. Next, put $R' = T_u^{-1}T_v^{-1}R$. This has $\psi(R') = R_{2\pi/3}$, so R' must be a rotation through $2\pi/3$ about some point. We have seen that $R(P) = P_1 = P + u + v$ so $R'(P) = P$ so P must be the centre of the rotation R', so $R' = R_{P,2\pi/3}$. Now suppose that $f \in I(X)$. Then f must send P to the centre of some motif, say $f(P) = nu + mv$

for some $n, m \in \mathbb{Z}$, so $T_v^{-m} T_u^{-n} f$ fixes P and preserves X. After multiplying by S if necessary we get a rotation that fixes P and preserves X, which must be a power of R'. As T_u, T_v, S and R' lie in $\langle R, S\rangle$ we deduce that $f \in \langle R, S\rangle$. Thus $I(X) = \langle R, S\rangle$ as required.

Q3: Let X be the pattern of hexagons shown below.



I claim that $I(X) = \langle S_l, S_m, S_n\rangle$. One checks directly that S_l, S_m and S_n send X to itself, so $\langle S_l, S_m, S_n\rangle \leq I(X)$. Clearly $S_m = S_0$ and $S_n = S_{-2\pi/6}$ and it follows that $\langle S_m, S_n\rangle = D_6$. Moreover, the $S_{l'} = S_\pi$ lies in D_6 so it can be written in terms of S_n and S_m (an explicit expression is $S_{l'} = S_n S_m S_n S_m S_n$). It follows that the map $T_u = S_{l'} S_l$ lies in $\langle S_l, S_m, S_n\rangle$. We also have $S_m S_n = R_{\pi/3}$ so the map $T_v = T_{R_{\pi/3} u} = R_{\pi/3} T_u R_{\pi/3}^{-1}$ also lies in $\langle T_l, T_m, T_n\rangle$. Given an arbitary element $g \in I(X)$ we see in the usual way that the map $h = T_u^{-n} T_v^{-m} g$ fixes O for some $n, m \in \mathbb{Z}$, and thus h lies in the symmetry group of the hexagon around O, which is the group $D_6 = \langle S_m, S_n\rangle$. It follows that the map $g = T_v^m T_u^n h$ lies in $\langle S_l, S_m, S_n\rangle$, which proves that $I(X) = \langle S_l, S_m, S_n\rangle$ as claimed.