Groups and Symmetry Problem Set 3 — Solutions

Q1: We have

$$G_{L,u/2}(0,0) = u/2 + S_L(0,0) = (1/2,0) + (0,\sqrt{3}/2) = v,$$

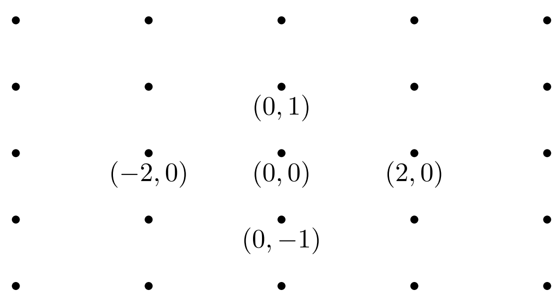
so the map $f = T_{-v}G_{L,u/2} = T_{u/2-v}S_L$ satisfies $f(0,0) = (0,0)$. As L is parallel to the x-axis we have $\psi(S_L) = S_0$ and $\psi(T_a) = 1$ for all a so $\psi(f) = S_0$. Thus $f(x) = S_0(x) + b$ for some b, but $f(0,0) = (0,0)$ so $b = (0,0)$ so $f = S_0$. Thus $G_{L,u/2} = T_v f = T_v S_0$, which writes $G_{L,u/2}$ in terms of $\{T_u, T_v, R_{\pi/3}, S_0\}$ as required.

Similarly, we have $R_{P,\pi}(0,0) = (1,0) = u$ so the map $h = T_{-u}R_{P,\pi}$ satisfies $h(0,0) = (0,0)$. It also has $\psi(h) = \psi(T_{-u})\psi(R_{P,\pi}) = R_\pi$ so we must have $h = R_\pi$, so $R_{P,\pi} = T_u R_\pi = T_u R^3_{\pi/3}$.

Q2: First, if $A \in O_2$ then $A = \psi(A)$. Thus, if $A \in H \cap O_2$ then $A = \psi(A) \in \psi(H)$, so $H \cap O_2 \leq \psi(H)$.

Now suppose that $L = \mathrm{Trans}(H)$; we must show that $\psi(H) \leq H \cap O_2$. If $A \in \psi(H)$ then there is an element $h \in H$ with $\psi(h) = A$, which means that $h(x) = Ax + a$ for some $a \in \mathbb{R}^2$. Next, note that $a = h(0)$, so $a \in L$ (by the definition of L). We are assuming that $L = \mathrm{Trans}(H)$, so $a \in \mathrm{Trans}(H)$, which means that $T_a \in H$. This means that the function $g = T_a^{-1}h$ also lies in H. Clearly $g(x) = h(x) - a = Ax$, so g corresponds to the element $A \in O_2$. This means that $A \in H \cap O_2$, as claimed.

Q3: The group $\mathrm{Trans}(H) \leq \mathbb{R}^2$ looks like this:



We know that if $A \in \psi(H)$ then $A.\mathrm{Trans}(H) = \mathrm{Trans}(H)$ (Lemma 5.8 in the notes). If A is a rotation about the origin that sends $\mathrm{Trans}(H)$ to itself, then the angle must be 0 or π, so $A = I$ or R_π. If A is the reflection across a line L through the origin, then L must be either the x-axis or the y-axis, so $A = S_0$ or S_π. Thus $\psi(H) \subseteq \{1, S_0, S_\pi, S_0S_\pi\}$ and so $|\psi(H)| \leq 4$.

Here is a slightly more formal argument. We know that $A.\mathrm{Trans}(H) = \mathrm{Trans}(H)$, so $A(0,1)$ lies in $\mathrm{Trans}(H)$ and has length 1, so $A(0,1) = (0,1)$ or $A(0,1) = (0,-1)$. In the first case we define $A_1 = A$, and in the second we define $A_1 = S_0A$; either way we have $A_1.\mathrm{Trans}(H) = \mathrm{Trans}(H)$ and $A_1(0,1) = (0,1)$. As A_1 preserves lengths and angles we see that $A_1(2,0)$ is perpendicular to $A_1(0,1) = (0,1)$ and $\|A_1(2,0)\| = 2$; the only possibilities are $A_1(2,0) = (2,0)$ or $A_1(2,0) = (-2,0)$. In the first case we put $A_2 = A_1$,and in the second we define $A_2 = S_\pi A_1$; either way we have $A_2(0,1) = (0,1)$ and $A_2(2,0) = (2,0)$. As $(0,1)$ and $(2,0)$ are a basis of $\mathbb{R}^2$, this means that $A_2 = 1$, and it follows that A is either 1, S_0, S_π or $S_0S_\pi = R_\pi$.

Q4:

(a) (1)⇒(2): Suppose G acts transitively and that $z \in X$. Then for any $y \in X$ there exists $g \in G$ such that $g * z = y$, by the definition of transitivity. This means that $y \in \{g * z \mid g \in G\} = G * z$. As every y lies in $G * z$, we have $G * z = X$, as required.

(2)⇒(3): if the condition $G * z = X$ holds for every element z of the nonempty set X, then it certainly holds for some element.

(3)⇒(1): Suppose that $G * z = X$ for some element $z \in X$. Let x and y be points of X. Then $x \in X = G * z$, so $x = a * z$ for some $a \in G$. Similarly $y = b * z$ for some $b \in G$. Thus the element $g = ba^{-1}$ satisfies $g * x = (ba^{-1}) * a * z = b * z = y$. Thus G acts transitively, as claimed.

(b) (1) Suppose $x, y \in \{1, \dots, n\}$. If $x = y$ let $\sigma \in S_n$ be the identity permutation, otherwise let σ be the transposition $(x\ y)$. Either way we have $\sigma * x = y$, so the action is transitive.

(2) This action is not transitive. The square X_4 contains the point $P = (1, 0)$, and also the point $Q = (1/2, 1/2)$ on the edge between $(1, 0)$ and $(0, 1)$. There is no element $g \in D_4$ such that $g(P) = Q$.

(3) This action is not transitive. To see this, put $x = (1\ 2\ 3)$ and $y = (1\ 2\ 3)(4\ 5\ 6)$. It is easy to see that $x^3 = y^3 = 1$, so x and y lie in the set under consideration. As x and y have different cycle types, they are not conjugate. More explicitly, for any $g \in S_6$ the permutation $g * x = gxg^{-1}$ satisfies $(gxg^{-1})(g(4)) = g(4)$, but there is no number i with $y(i) = i$, so $g * x \neq y$.

(c) Given any two elements $xH, yH \in G/H$, the element $g = yx^{-1}$ satisfies $g * (xH) = yH$; this shows that G acts transitively.

(d) Suppose that G acts transitively on X. Choose a point $a \in X$ and put $H = \{g \in G \mid g * a = a\}$. Define $\phi\colon G/H \to X$ by $\phi(xH) = x * a$. To see that this is well-defined, note that if $xH = yH$ then $x^{-1}y \in H$ so $(x^{-1}y) * a = a$ so $x * a = x * (x^{-1}y) * a = y * a$. Conversely, if $\phi(xH) = \phi(yH)$ then $x * a = y * a$ so $a = (x^{-1}y) * a$ so $x^{-1}y \in H$ so $xH = yH$; this shows that ϕ is injective. Moreover, for any $b \in X$, there is an element $x \in G$ with $x * a = b$, because the action is transitive, and so $\phi(xH) = b$. This shows that ϕ is surjective and thus bijective. We also have

$$\phi(g * (xH)) = \phi(gxH) = (gx) * a = g * (x * a) = g * \phi(xH),$$

so ϕ gives an equivalence between the two actions.