Groups and Symmetry Problem Set 4 — Solutions

Q1:

(a) It is clear that g is linear, and we have

$$\|g(x,y,z)\|^2 = y^2 + z^2 + x^2 = x^2 + y^2 + z^2 = \|(x,y,z)\|^2,$$

so g preserves lengths. Thus $g \in O_3$.

(b) Visibly $g(a,a,a) = (a,a,a)$ for any a. To get a unit vector, put $a = 1/\sqrt{3}$.

(c) We have $g^2(x,y,z) = g(y,z,x) = (z,x,y)$ and $g^3(x,y,z) = g(z,x,y) = (x,y,z)$, so $g^3 = 1$ and g has order 3. Because $g \in O_3$ we have $\det(g) = \pm 1$ and $\det(g)^3 = \det(g^3) = \det(1) = 1$ which would give a contradiction if $\det(g)$ were -1, so $\det(g) = 1$. Thus $g \in SO_3$.

(d) The vertices of the standard cube are the points (x,y,z) for which $x,y,z \in \{1,-1\}$. Clearly, if (x,y,z) satisfies this condition then so does (y,z,x), so g carries vertices of the cube to vertices of the cube, so it sends the cube to itself.

(e) As $g \in SO_3$, it must be a rotation. Part (b) shows that the axis of rotation is the line $x = y = z$. Part (c) implies that the angle of rotation is $2\pi/3$. As $g(0,0,1) = (0,1,0)$ we see that g carries the z-axis to the y-axis and thus that the direction of rotation (as seen while looking from $(1,1,1)$ towards the origin) is clockwise.

Q2:

(a) A twist of one third around the z-axis sends v_2 to v_3, v_3 to v_4 and v_4 to v_2. Such a twist preserves horizontal planes, and v_2 lies in the plane $z = -b$ so the same is true of v_3 and v_4.

(b) Let w_2, w_3 and w_4 be the points in the xy plane lying above v_2, v_3 and v_4. Then $w_2 = (a,0)$ and w_3 and w_4 are obtained from w_2 by rotating around the origin through $2\pi/3$ in either direction, so the coordinates are $(\cos(\pm 2\pi/3)a, \sin(\pm 2\pi/3)a)$, which is equal to $(-a/2, \pm a\sqrt{3}/2)$. In combination with (a) this means that $v_3 = (-a/2, a\sqrt{3}/2, -b)$ and $v_3 = (-a/2, -a\sqrt{3}/2, -b)$.

(c) All the vertices of a tetrahedron have the same distance from the centre, so $\|v_2\|^2 = \|v_1\|^2$, or in other words $a^2 + b^2 = 3$. The distance between any two vertices is the same, so $d(v_1, v_2) = d(v_3, v_4)$. We have $v_1 - v_2 = (-a, 0, b + \sqrt{3})$ so $d(v_1, v_2)^2 = a^2 + (b + \sqrt{3})^2$. We also have $v_2 - v_3 = (0, a\sqrt{3}, 0)$, so $d(v_2, v_3)^2 = 3a^2$. It follows that $a^2 + (b + \sqrt{3})^2 = 3a^2$ as claimed.

(d) Our second equation expands out to give $b^2 + 2b\sqrt{3} + 3 - 2a^2 = 0$. Our first equation gives $a^2 = 3 - b^2$ and after substituting this in we get $3b^2 + 2b\sqrt{3} - 3 = 0$, so $b^2 + 2b/\sqrt{3} - 1 = 0$, so $(b + 1/\sqrt{3})^2 = 4/3$. As b must clearly be positive this gives $b = 1/\sqrt{3}$. This implies that $a^2 = 3 - b^2 = 3 - 1/3 = 8/3$ so $a = \sqrt{8/3} = 2\sqrt{2/3}$. Putting this back into our equations for the v_i gives

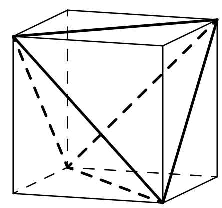
$$\begin{aligned}
v_1 &= (0, 0, \sqrt{3}) \\
v_2 &= (2\sqrt{2/3}, 0, -1/\sqrt{3}) \\
v_3 &= (-\sqrt{2/3}, \sqrt{2}, -1/\sqrt{3}) \\
v_4 &= (-\sqrt{2/3}, -\sqrt{2}, -1/\sqrt{3}).
\end{aligned}$$

(e) Let V be the matrix whose columns are v_1, v_2, v_3 and v_4, so gV is the matrix whose columns are gv_1, gv_2, gv_3 and gv_4. We have

$$gV = \begin{pmatrix} 1/3 & 0 & 2\sqrt{2}/3 \\ 0 & -1 & 0 \\ 2\sqrt{2}/3 & 0 & -1/3 \end{pmatrix} \begin{pmatrix} 0 & 2\sqrt{2/3} & -\sqrt{2/3} & -\sqrt{2/3} \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \\ \sqrt{3} & -1/\sqrt{3} & -1/\sqrt{3} & -1/\sqrt{3} \end{pmatrix}$$
$$= \begin{pmatrix} 2\sqrt{2/3} & 0 & -\sqrt{2/3} & -\sqrt{2/3} \\ 0 & 0 & -\sqrt{2} & \sqrt{2} \\ -1/\sqrt{3} & \sqrt{3} & -1/\sqrt{3} & -1/\sqrt{3} \end{pmatrix},$$

so $g(v_1) = v_2$, $g(v_2) = v_1$, $g(v_3) = v_4$ and $g(v_4) = v_3$. Thus, the permutation associated to g is $(1\ 2)(3\ 4)$.

Q3: The following picture shows a tetrahedron embedded inside a cube. We'll assume as usual that the edges of the cube have length 2.



Alternatively, we can say that the cube is built from the tetrahedron by attaching a pyramid to each face. All the pyramids have the same shape: the base edges have length $2\sqrt{2}$, and the remaining edges have length 2. Thus, any isometry of the tetrahedron carries pyramids to pyramids and thus gives an isometry of the cube. This shows that $\mathrm{Dir}(\mathrm{Tet}) \leq \mathrm{Dir}(\mathrm{Cube})$.

Q4:

(a) For any $g \in \mathrm{Dir}(\mathrm{Cube})$ we let $\psi(g)$ be the permutation such that $g(M_i) = M_{\sigma(i)}$ for $i = 1, 2, 3$.

(b) Let g be a half turn around the vector $(0, 1, 1)$, let h be a one-third turn anticlockwise about the vector $(1, 1, 1)$, and let k be a quarter turn anticlockwise about the vector $(0, 0, 1)$. Then $\psi(g) = (2\ 3)$, $\psi(h) = (1\ 2\ 3)$ and $\psi(k) = (1\ 2)$.

(c) Part (b) shows that the image of ψ contains $(1\ 2)$ and $(2\ 3)$, and these two transpositions generate S_3 so ψ is surjective. More explicitly, we have

$$\begin{array}{ll} \psi(1) = 1 & \psi(k) = (1\ 2) \\ \psi(h) = (1\ 2\ 3) & \psi(g) = (2\ 3) \\ \psi(h^{-1}) = (1\ 3\ 2) & \psi(gh) = (1\ 3). \end{array}$$

(d) Let G be the group of matrices of the form

$$g = \begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{pmatrix}$$

such that $\epsilon_1\epsilon_2\epsilon_3 = 1$. It is easy to see that G is a subgroup of SO_3 and that it preserves the cube so it is a subgroup of $\mathrm{Dir}(\mathrm{Cube})$. If $g \in G$ and $x \in \mathbb{R}$ then $g(x, 0, 0) = (\pm x, 0, 0)$, so g preserves the x-axis, or in other words $g(M_1) = M_1$. Similarly, we have $g(M_2) = M_2$ and $g(M_3) = M_3$, so $\psi(g) = 1$, so $G \leq \ker(\psi)$.

Conversely, if $g \in \ker(\psi)$ then $g(M_1) = M_1$. The axis M_1 meets the surface of the cube at $(1, 0, 0)$ and $(-1, 0, 0)$, so $g(1, 0, 0) = \epsilon_1(1, 0, 0)$ for some $\epsilon_1 \in \{1, -1\}$. Similarly we have $g(0, 1, 0) = \epsilon_2(0, 1, 0)$ and $g(0, 0, 1) = \epsilon_3(0, 0, 1)$ for some $\epsilon_2, \epsilon_3 \in \{1, -1\}$. Thus, the matrix of g has the form described above, and as $g \in \mathrm{Dir}(\mathrm{Cube}) \leq SO_3$ we have $\det(g) = 1$ so $\epsilon_1\epsilon_2\epsilon_3 = 1$ so $g \in G$. This shows that $\ker(\psi) = G$.

We can define an isomorphism $\chi\colon C_2 \times C_2 = \{\pm 1\} \times \{\pm 1\} \to G$ by $\chi(\epsilon_1, \epsilon_2) = (\epsilon_1, \epsilon_2, \epsilon_1\epsilon_2)$.