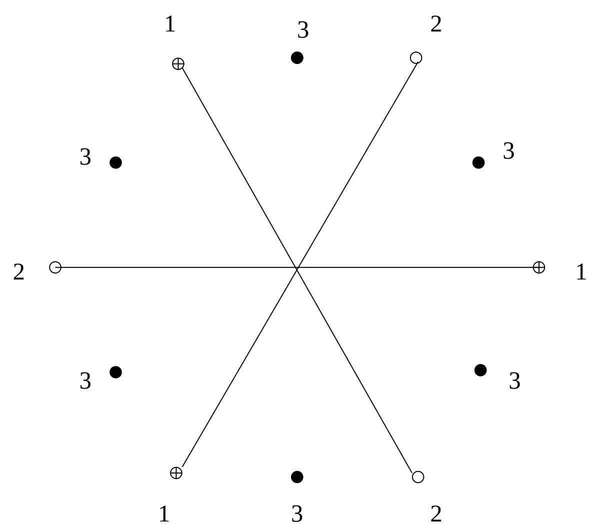
Groups and Symmetry Problem Set 5 — Solutions

Q1: Put $G = \mathrm{Dir}(X)$, which is a finite subgroup of SO_3. The normalisations of the midpoints of the edges are poles of degree 2, the normalisations of the centres of the triangular faces are poles of degree 3, and the normalisations of the centres of the square faces are poles of degree 4. The only one of the standard groups that has poles of degrees 2, 3 and 4 is G_2, so G must be conjugate to G_2 and thus isomorphic to S_4.

One can see from the picture that multiplication by -1 preserves X and so Proposition 6.9 in the notes tells us that $\mathrm{Symm}(X) \simeq \{\pm 1\} \times \mathrm{Dir}(X) \simeq \{\pm 1\} \times S_4$.

Q2: Let S be the surface of the cube. For any point $x \in S$ we have $|Gx| = |G|/|\mathrm{stab}_G(x)| = 24/|\mathrm{stab}_G(x)|$. Moreover, $\mathrm{stab}_G(x)$ is the group of all rotations around x that preserve the cube. For most points x there are no such rotations (except for the identity) and so $|\mathrm{stab}_G(x)| = 1$ and the orbit Gx has order 24. If x is the centre of a face then $\mathrm{stab}_G(x)$ is cyclic of order 4 and so $|Gx| = 24/4 = 6$. In fact, Gx consists of the centres of the 6 faces. If x is a vertex of the cube then $\mathrm{stab}_G(x)$ is cyclic of order 3 and so $|Gx| = 24/3 = 8$. In fact, in this case Gx consists of the 8 vertices of the cube. If x is the midpoint of an edge then $\mathrm{stab}_G(x)$ has order 2 and so $|Gx| = 24/2 = 12$. In fact, in this case Gx consists of the midpoints of the 12 edges of the cube. In all other cases we have $|Gx| = 24$.

Q3: The points marked 1 form an orbit of size 3, the points marked 2 form another orbit of size 3, and the points marked 3 form an orbit of size 6. Thus, there are 3 orbits altogether.



Recall that

$$D_3 = \{1, R_{2\pi/3}, R_{4\pi/3}, S_0, S_{2\pi/3}, S_{4\pi/3}\}.$$

The identity element of D_3 fixes all 12 points of X. The rotations $R_{2\pi/3}$ and $R_{4\pi/3}$ have no fixed points in X. The axis of the reflection S_0 passes through 2 of the points of X. These two points are fixed under S_0, and the remaining points are not. Similarly, the reflections $S_{2\pi/3}$ and $S_{4\pi/3}$ have two fixed points each. Thus

$$\frac{1}{|G|} \sum_{g \in G} |\,\mathrm{Fix}(g)| = \frac{1}{6}(12 + 0 + 0 + 2 + 2 + 2) = 3,$$

which is equal to the number of orbits, as predicted by the orbit counting theorem.

Q4: Let the sizes of the 4 orbits be d_1, d_2, d_3 and d_4 with $d_1 \leq d_2 \leq d_3 \leq d_4$. These sizes must divide 77, and thus must be 1, 7, 11 or 77. We must also have $d_1 + d_2 + d_3 + d_4 = 96$. If $d_4 \neq 77$ then $d_i \leq 11$ for all i so $96 = d_1 + d_2 + d_3 + d_4 \leq 4 \times 11 = 44$, which is false. Thus $d_4 = 77$ and $d_1 + d_2 + d_3 = 96 - 77 = 19$. By similar arguments or by inspection we must have $d_1 = 1$, $d_2 = 7$ and $d_3 = 11$. Thus, there is precisely one orbit of size 1. If this orbit has the form $\{x\}$ then x is fixed under the action of G. If y is in any of the other orbits then $|Gy| > 1$ so y is not fixed. Thus, there is precisely one fixed point.

Q5: A_5 consists of elements of the following types:

- the identity, with order 1
- transposition pairs (such as (1 2)(3 4)), with order 2
- 3-cycles (such as (1 2 3)), with order 3
- 5-cycles (such as (1 2 3 4 5)) with order 5.

There are thus no elements of order 15.

Now let G be a subgroup of A_5 with $|G| = 30$. The group A_5 is isomorphic to the subgroup G_3 of SO_3, so G is isomorphic to some subgroup of G_3 and thus to a finite subgroup of SO_3. It follows by the classification that G is isomorphic to G_1, G_2 or G_3, or to $\widetilde{C}_n$ or $\widetilde{D}_n$ for some n. As $|G| = 30$ which is different from the orders of G_1, G_2 and G_3 we must have $G \simeq \widetilde{C}_{30}$ or $G \simeq \widetilde{D}_{15}$. However $\widetilde{C}_{30}$ and $\widetilde{D}_{15}$ both contain elements of order 15 and G does not, which gives a contradiction. Thus there can be no subgroups of A_5 of order 30.