Groups and Symmetry — Examinable Proofs

All definitions are examinable. All results and examples in the notes and problem sheets may provide insight that you will need to solve problems in the exam. The results listed below are the ones for which you may be asked to give a precise statement or a proof (including any supporting lemmas or definitions). You should also remember the following formulae in O_2:

$$R_\theta R_\phi = R_{\theta+\phi} \qquad S_\theta S_\phi = R_{\theta-\phi}$$
$$R_\theta S_\phi = S_{\theta+\phi} \qquad S_\theta R_\phi = S_{\theta-\phi}.$$

Proposition 1.5. *O_n is a subgroup of GL_n.*

Theorem 1.13. *Any matrix $A \in SO_2$ has the form R_θ for some θ. Any matrix $A \in O_2 \setminus SO_2$ has the form S_θ for some θ.*

Proposition 1.17. *For any $X \subseteq \mathbb{R}^n$ and $A \in O_n$ we have $\mathrm{Symm}(AX) = A\,\mathrm{Symm}(X)A^{-1}$ and $\mathrm{Dir}(AX) = A\,\mathrm{Dir}(X)A^{-1}$.*

Proposition 2.3. *Let G be a finite subgroup of SO_2. Then $G = C_n$ for some n.*

Theorem 2.4. *Let G be a finite subgroup of O_2. Then either $G = C_n = \mathrm{Dir}(X_n)$ for some n, or $G = R_\theta D_n R_\theta^{-1} = \mathrm{Symm}(R_\theta X_n)$ for some n and θ.*

Proposition 3.6. *The map ψ is a surjective homomorphism with kernel Trans_n, and thus it induces an isomorphism $\mathrm{Isom}_n / \mathrm{Trans}_n \simeq O_n$. Moreover, $\det\colon \mathrm{Isom}_n \to \{\pm 1\}$ is also a homomorphism.*

Proposition 3.7. *For any $f \in \mathrm{Isom}_n$ and $b \in \mathbb{R}^n$ we have $fT_bf^{-1} = T_{\psi(f)a}$.*

Proposition 4.1. *For any $f \in \mathrm{Isom}_2$, precisely one of the following holds:*

(a) $f = 1$
(b) $f = T_a$ *for some* $a \in \mathbb{R}^2 \setminus \{0\}$
(c) $f = R_{\theta,a}$ *for some* $a \in \mathbb{R}^2$ *and* $\theta \in (0, 2\pi)$
(d) $f = S_L$ *for some line* $L < \mathbb{R}^2$
(e) $f = G_{L,b}$ *for some* L *and some nonzero vector* b *parallel to* L.

Theorem 4.3. *Let H be a subgroup of Isom_2, and suppose that H contains no translations (other than the trivial translation $T_0 = 1$). Then there is a point $a \in \mathbb{R}^2$ such that $f(a) = a$ for all $f \in H$, and thus $H \leq T_a O_2 T_a^{-1}$.*

You should learn the structure of the proof, but you need not remember the detailed formulae.

Proposition 5.4. $\mathrm{Isom}(X)$ *is generated by …*

You should be able to prove this kind of result for any wallpaper pattern that you might be given.

Theorem 5.10. *Let H be a wallpaper group. Then the rotational order of H is 1, 2, 3, 4 or 6.*

Proposition 6.10. *If $G \leq O_3$ and $-1 \in G$ and $H = G \cap SO_3$ then $G = H \times \{\pm 1\}$.*

Theorem 6.12. *The homomorphism $\phi\colon \mathrm{Symm}(\mathrm{Tet}) \to S_4$ (obtained from the action on the vertices) is an isomorphism, and it also gives an isomorphism $\mathrm{Dir}(\mathrm{Tet}) \to A_4$.*

Theorem 6.15. *The homomorphism $\phi\colon \mathrm{Dir}(\mathrm{Cube}) \to S_4$ (obtained from the action on the set of long diagonals) is an isomorphism.*

Proposition 7.1. *The group $\mathrm{Symm}(\mathrm{Oct})$ is the same as $\mathrm{Symm}(\mathrm{Cube})$ (and thus is isomorphic to $S_4 \times \{\pm 1\}$).*

Theorem 11.1. *Let G be a finite subgroup of SO_3. Then G is conjugate to one of the groups G_1, G_2, G_3, $\widetilde{C}_n$ or $\widetilde{D}_n$ (for some n).*

You should be able to state this precisely with all supporting definitions, but you will not be asked to prove it.

Lemma 11.7. *The action of G on $\mathbb{R}^3$ preserves the set P of poles. Moreover, if $v \in P$ and $g \in G$ then gv has the same order as v.*
This is a slightly strengthened version of Lemma 11.7 in the notes; a proof was given in lectures.

Abstract Group Theory

Theorem 1.1. (a) *There is at least one Sylow p-subgroup, so $n_p > 0$.*
(b) *Moreover, n_p divides m and is congruent to 1 mod p.*
(c) *Any two Sylow p-subgroups are conjugate.*
(d) *Any p-subgroup of G is contained in a Sylow p-subgroup.*

You should be able to state this precisely with all supporting definitions and notation. The proof is not examinable.

Lemma 1.4. *Let P be a finite p-group, and let X be a set with an action of P. Put*

$$\mathrm{Fix}(P) = \mathrm{Fix}(P, X) = \{x \in X \mid gx = x \text{ for all } g \in P\}.$$

Then $|\mathrm{Fix}(P)| = |X| \pmod p$. In particular, if $|X| \neq 0 \pmod p$ then $\mathrm{Fix}(P) \neq \emptyset$.

Proposition 2.6. *If P is a nontrivial p-group then $Z(P) \neq \{1\}$.*

Lemma 2.7. *Let G be a finite group, and let P and Q be subgroups of G. Define a function $\phi\colon P \times Q \to G$ by $\phi(x, y) = xy$.*
(a) *If every element of P commutes with every element of Q, then ϕ is a homomorphism.*
(b) *If we also have $P \cap Q = \{1\}$, then ϕ is injective.*

Proposition 2.9. *Let G be a finite group, and let P and Q be* normal *subgroups of orders p and q. Suppose that p and q are coprime, and that $pq = |G|$. Then $G \simeq P \times Q$.*

Proposition 2.10. *Let G be a group of order pq where p and q are primes and $p < q$. Suppose also that $q - 1$ is not divisible by p. Then $G \simeq C_p \times C_q$.*