NOTES FOR PART III HOMOTOPY THEORY

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1. Introduction

Let X and Y be spaces with given basepoints 0_X, 0_Y. We say that two based maps $f_0, f_1\colon X \to Y$ are *homotopic* if there is a continuous family of based maps $f_t\colon X \to Y$ (for $0 \leq t \leq 1$) interpolating between them. We write $[X,Y]$ for the set of homotopy classes of based maps, and $\pi_n(X)$ for $[S^n, X]$. This has a natural group structure when $n > 0$, which is Abelian when $n > 1$.

It turns out that it is often possible to determine the set $[X,Y]$, and that this is a rather useful and interesting thing to do.

Example 1.1.

(1) By a theorem of Serre, we have

$$\mathbb{Q} \otimes \pi_n S^m = \begin{cases} \mathbb{Q}\iota_n & \text{if } n = m \\ \mathbb{Q}w_m & \text{if } m \text{ is odd and } n = 2m-1 \end{cases}$$

Here $\iota_n\colon S^n \to S^n$ is the identity map, and $w_m\colon S^{2m-1} \to S^m$ is a certain (fairly simple) map defined by Whitehead.

(2) The groups $\pi_n S^m$ themselves (before tensoring with $\mathbb{Q}$) are known for a fairly wide range of n and m, but the answers are complicated. For example, $\pi_3 S^2 = \mathbb{Z}$, generated by the Hopf map $\nu\colon S^3 \to S^2$. To define this, think of S^3 as $\{(z,w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\}$ and S^2 as $\mathbb{C} \cup \{\infty\}$, and then $\nu(z,w) = z/w$.

(3) For all closed surfaces M except S^2 and $\mathbb{R}P^2$, we have $\pi_n M = 0$ for $n > 1$. If M is an oriented surface of genus g then $\pi_1 M$ is generated by $2g$ generators $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$ modulo one relation

$$[\alpha_1, \beta_1][\alpha_2, \beta_2] \ldots [\alpha_g, \beta_g] = 1.$$

(4) If $O = \bigcup_n O(n)$ is the infinite orthogonal group then $\pi_* O$ is periodic of period 8 (this is called *Bott periodicity.*) The first eight groups are

$$\pi_* O = (\mathbb{Z}/2, \mathbb{Z}/2, 0, \mathbb{Z}, 0, 0, 0, \mathbb{Z}, \ldots).$$

We next discuss some applications of these ideas.

The most obvious kind of application of homotopy theory is as an approximation to geometry. One would like to classify smooth manifolds (for example) up to diffeomorphism. The first step is to classify them up to homotopy equivalence, and then perhaps up to homeomorphism, and then up to diffeomorphism. It turns out that two-dimensional closed manifolds are diffeomorphic iff they are homotopy equivalent, and similarly for large classes of three-dimensional manifolds.

There are also many more indirect applications to geometry. For example, suppose that we have a smooth manifold M (which for technical reasons should have dimension at least five) and we want to know how many other smooth structures there are on the same underlying topological space (up to a suitable kind of equivalence). It turns out that the answer involves the calculation of certain groups of homotopy classes such as $[M, \mathrm{Top}/\mathrm{Diff}]$ for a certain space Top/Diff. A great deal is known about the homology and homotopy groups of spaces such as this.

For another example, we explain the basic ideas of bordism theory. We say that two closed manifolds M_0, M_1 of dimension k are *cobordant* if there is a manifold W of dimension $(k+1)$ whose boundary is diffeomorphic to $M_0 \amalg M_1$. We write MO_k for the set of cobordism classes of closed k-manifolds. These sets form a graded $\mathbb{F}_2$-algebra under disjoint union and cartesian product. A fundamental theorem of Thom shows that there are spaces $MO(N)$ such that $MO_k = \pi_{k+N} MO(N)$ for $N \gg 0$. It is easy to compute the homology groups of $MO(N)$ (with mod 2 coefficients), and one can use a tool called the Adams spectral sequence to compute the groups MO_* from this. The answer is that

$$MO_* = \mathbb{F}_2[x_2, x_4, x_5, x_6, x_8, \dots],$$

a polynomial algebra over $\mathbb{F}_2$ with one generator x_n in each degree n not of the form $2^k - 1$.

As well as these applications to geometry, there are also applications to pure algebra. For example, for any group G there is a space BG (which is essentially unique up to homotopy) such that $\pi_1 BG = G$ and $\pi_k BG = 0$ for $k \neq 1$. The homology and cohomology of BG carry a lot of useful information about the group G. For many popular groups G there is a simple geometric model of BG; for example, $B\Sigma_n$ is the set of subsets of order n in a vector space of countable dimension over $\mathbb{R}$, suitably topologised.

For another example, we mention the basic ideas about algebraic K theory. If R is a commutative ring then $K_0(R)$ is the set of formal expressions $P - Q$, where P and Q are finitely-generated projective modules and $P - Q$ is identified with $P' - Q'$ whenever $P \oplus Q' \oplus M$ is isomorphic to $P' \oplus Q \oplus M$ for some M. This is a useful and basic invariant of the ring R. There are similarly explicit definitions of groups $K_1(R)$ and $K_2(R)$, which fit into various exact sequences of a few terms. For a long time it was not understood how to define $K_d(R)$ for $d > 2$ so as to get long exact sequences and effective methods of calculation. This problem was eventually solved when Quillen showed how to construct a space $K(R)$ such that $K_d(R) = \pi_d K(R)$ for $d < 3$. This made it natural to define $K_d(R) = \pi_d K(R)$ for all d, and the resulting theory has been very successful.

2. Some constructions with spaces

We work everywhere with compactly generated weak Hausdorff spaces. This sometimes means using a topology on product spaces that is slightly different from the usual one. You can fairly safely ignore all this, or consult the Appendix for details if you prefer. The word "map" will always mean a continuous function, but the word "function" does not imply continuity.

Our results involving (co)homology are valid as stated for singular (co)homology, but may need additional hypotheses if we use the Alexander-Spanier version.

We will often use pullbacks and pushouts of spaces. Given maps $X \xrightarrow{h} Z \xleftarrow{k} Y$, we define their *pullback* to be the space

$$X \times_Z Y = \{(x, y) \in X \times Y \mid h(x) = k(y)\}.$$

This is topologised as a (closed) subspace of $X \times Y$. For any space W, maps $W \to X \times_Z Y$ biject in an evident way with commutative squares of the form

$$\begin{tikzcd} W \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "h"] \\ Y \arrow[r, "k"'] & Z. \end{tikzcd}$$

If the map $W \to X \times_Z Y$ arising from such a square is a homeomorphism, we say that the square is a *pullback square.* We also say that g is the pullback of h along k.

Before defining pushouts, we make a few remarks about equivalence relations. A relation on a space X can be thought of as a subset of $X \times X$. Given a relation R on X, we can take the intersection of all equivalence relations that contain R and are closed as subsets of $X \times X$; call this relation E. It is easy to check that E is an equivalence relation and is closed as a subspace of $X \times X$; clearly it is the smallest such. We can then form the quotient space X/E. One can show that for any space Y, the maps $f\colon X/E \to Y$ biject with maps $f'\colon X \to Y$ such that aRb implies $f'(a) = f'(b)$. The restriction to closed equivalence relations comes from the fact that we only consider weak Hausdorff spaces.

Now suppose that we start with maps $Y \xleftarrow{g} W \xrightarrow{f} X$. To define the *pushout* $X \cup_W Y$, we start with the disjoint union $X \amalg Y$ and impose the smallest possible closed equivalence relation such that $f(w) \in X$ is identified with $g(w) \in Y$ for all $w \in W$. It follows that for any space Z, the maps $X \cup_W Y \to Z$ biject with commutative squares of the form

$$\begin{tikzcd} W \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "h"] \\ Y \arrow[r, "k"'] & Z. \end{tikzcd}$$

If the map $X \cup_W Y \to Z$ arising from such a square is a homeomorphism, we say that the square is a *pushout square.* We also say that k is the pushout of f along g.

If f is a closed inclusion, it is not hard to produce a (discontinuous) bijection $X \cup_W Y = (X \setminus fW) \amalg Y$.

We will also need to consider the space

$$C(X,Y) = \{ \text{ continuous maps } u\colon X \to Y\}.$$

As a first attempt to topologise this, let K be a compact subset of X and U an open subset of Y, and define

$$W(K,U) = \{u \in C(X,Y) \mid u(K) \subseteq U\}.$$

The collection of finite intersections $W(K_1,U_1) \cap \ldots \cap W(K_n,U_n)$ forms a basis for a topology on $C(X,Y)$, called the compact-open topology. After modifying this topology to make it compactly generated, we get what we shall call the *standard topology* on $C(X,Y)$.

There are a number of natural maps that we need to consider, such as the composition map $\mathrm{comp}\colon C(Y,Z) \times C(X,Y) \to C(X,Z)$ defined by $\mathrm{comp}(u,v) =$

$u \circ v$, or the evaluation map $\mathrm{eval}\colon C(X,Y) \times X \to Y$ defined by $\mathrm{eval}(f,x) = f(x)$. Provided that we use the standard topologies, these are all continuous.

We also have an adjunction homeomorphism

$$\mathrm{adj}\colon C(X \times Y, Z) \to C(X, C(Y,Z))$$

defined by $\mathrm{adj}(f)(x)(y) = f(x,y)$. We will usually use the notation $f^\#$ for $\mathrm{adj}(f)$.

3. Smashouts

We next define the smashout of a pair of maps with common domain. It turns out that the smashout of two cofibrations is a cofibration, which is a key fact for the smooth development of the theory.

Definition 3.1. Given maps $u\colon A \to B$ and $v\colon C \to D$, we write

$$S(u,v) = \{(f,g) \in C(A,C) \times C(B,D) \mid vf = gu\}.$$

This is a closed subspace of $C(A,C) \times C(B,D)$. It can be thought of as the space of commutative squares of the form

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "u"'] & C \arrow[d, "v"] \\
B \arrow[r, "g"'] & D.
\end{tikzcd}$$

There is also a pullback diagram

$$\begin{tikzcd}
S(u,v) \arrow[r] \arrow[d] & C(A,C) \arrow[d, "v_*"] \\
C(B,D) \arrow[r, "u^*"'] & C(A,D),
\end{tikzcd}$$

so we can write $S(u,v) = C(A,C) \times_{C(A,D)} C(B,D)$.

Definition 3.2. Given maps $u\colon A \to B$ and $v\colon C \to D$, we define a space $P = (B \times C) \cup_{(A \times C)} (A \times D)$ by the following pushout diagram:

$$\begin{tikzcd}
A \times C \arrow[r, "1\times v"] \arrow[d, "u\times 1"'] & A \times D \arrow[d] \\
B \times C \arrow[r] & P.
\end{tikzcd}$$

There is an evident map $u\square v\colon P \to B \times D$, given by $u \times 1$ on $A \times D$ and by $1 \times v$ on $B \times C$; we call this the *smashout* of u and v. Given another map $w\colon E \to F$, we have a map $F(v,w)\colon C(D,E) \to S(v,w)$ defined by $F(u,v)(h) = (hv, wh)$. We cll this the *crossmap* of v and w.

Proposition 3.3. Let z be the unique map from the empty set to the set $\{0\}$. Then we have

$$\begin{aligned} z\square u &= u \\ u\square v &= v\square u \\ u\square(v\square w) &= (u\square v)\square w \\ F(z,u) &= u \\ F(u,F(v,w)) &= F(u\square v,w) \\ S(u,v) &= S(z,F(u,v)) \\ S(u,F(v,w)) &= S(u\square v,w) \end{aligned}$$

Definition 3.4. Suppose we have two maps $u\colon A\to B$ and $v\colon C\to D$. We say that u is *left orthogonal* to v (or that v is *right orthogonal* to u) if for every commutative square as shown (without the h) there exists a map h making the whole diagram commute.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"'] & C \arrow[d, "v"] \\ B \arrow[r, "g"'] \arrow[ur, "h"] & D. \end{tikzcd}$$

(We say that h *fills in the square.*) It is equivalent to say that the crossmap $F(u,v)\colon C(B,C)\to S(u,v)$ is surjective. Given a class M of maps, we write ${}^{\perp}M$ for the class of maps that are left orthogonal to every map in M, and $M^{\perp}$ for the class of maps that are right orthogonal to every map in M.

Exercise 3.5. If we work with sets rather than spaces, show that $\{\text{injections}\}^{\perp}=\{\text{surjections}\}$ and $\{\text{injections}\}={}^{\perp}\{\text{surjections}\}$.

Example 3.6. The map $\mathrm{Cyl}(j)\to\mathrm{Cyl}(Y)$ in the definition of a cofibration is just the smashout of the inclusion $\{1\}\rightarrowtail I$ with j. The smashout of the inclusion $\{0\}\rightarrowtail\mathbb{R}$ with itself is just the inclusion of $\{(x,y)\mid xy=0\}$ in $\mathbb{R}^2$.

4. POINTED SPACES

Definition 4.1. A *pointed space* is a space X with a given basepoint $0=0_X\in X$. We say that X is *well-pointed* or that it has a *non-degenerate basepoint* if the inclusion $\{0\}\rightarrowtail X$ is a cofibration. We will almost always assume that basepoints are nondegenerate.

Definition 4.2. We say that a map $f\colon X\to Y$ is *based* or *pointed* if $f(0_X)=0_Y$. We write $F(X,Y)$ for the space of pointed maps from X to Y.

Definition 4.3. Given a space X, we write X_+ for the pointed space $X\amalg\{0\}$. Given a closed subspace $Y\subseteq X$, we let X/Y denote the quotient space $X_+/\sim$ in which each point of Y is identified with 0. Given two pointed spaces X and Y, we define

$$X\vee Y=(X\amalg Y)/\{0_X,0_Y\}.$$

This is called the *wedge* of X and Y; it can also be identified with the subspace $X\times\{0_Y\}\cup\{0_X\}\times Y\subseteq X\times Y$. The *smash product* of X and Y is defined as

$$X\wedge Y=(X\times Y)/(X\vee Y).$$

Example 4.4. There is a homeomorphism $S^n \wedge S^m = S^{n+m}$. One way to see this is to think of S^1 as $I/\{0,1\}$ and show that the smash product of n copies of S^1 is $I^n/\partial(I^n)$; it is not hard to see that this is homeomorphic to S^n. Another way is to think of S^1 as the one-point compactification of $\mathbb{R}$, and to show that $S^1 \wedge \ldots \wedge S^1$ is the one-point compactification of $\mathbb{R}^n$, which is S^n.

We take our first picture of S^1 as the official definition.

Definition 4.5. We give $I = [0,1]$ the basepoint 0, and we define $S^1 = I/(0 \sim 1)$. We define

$$\begin{aligned} CX &= I \wedge X \\ \Sigma X &= S^1 \wedge X \\ PX &= F(I, X) \\ \Omega X &= F(S^1, X). \end{aligned}$$

These are called the cone on X and the suspension, path space, and loop space of X, respectively.

5. Fibrations and cofibrations

Definition 5.1. A map $f\colon X \to Y$ is a *homotopy equivalence* iff there is a map $g\colon Y \to X$ such that $fg \simeq 1_Y$ and $gf \simeq 1_X$. We write heq for the class of homotopy equivalences. We say that X and Y are *homotopy equivalent*, or that they have the same *homotopy type* if there exists a homotopy equivalence between them.

Definition 5.2. Let $j\colon X \to Y$ be a map. Define $\mathrm{Cyl}(X) = I \times X$, and let $i_1\colon X \to \mathrm{Cyl}(X)$ be the map $i_1(x) = (0,x)$. Let $\mathrm{Cyl}(j) = (I \times X) \cup_X Y$ be the following pushout:

$$\begin{tikzcd} X \arrow[r, "i_1"] \arrow[d, "j"'] & \mathrm{Cyl}(X) \arrow[d] \\ Y \arrow[r] & \mathrm{Cyl}(j). \end{tikzcd}$$

This is called the *mapping cylinder* of i. There is an evident map $k\colon (I \times X) \cup_X Y \to I \times Y$, which maps $I \times X$ by $1 \times j$ and Y by i_1. We say that j is a *cofibration* iff there is a map $r\colon I \times Y \to (I \times X) \cup_X Y$ such that $rk = 1$. This means that $I \times Y$ can be pushed down continuously onto the subspace $(I \times X) \cup_X Y$.

A cofibration which is a homotopy equivalence is called an *acyclic cofibration.* We write cof and acf for the classes of cofibrations and acyclic cofibrations.

Remark 5.3. The map $k\colon \mathrm{Cyl}(j) \to \mathrm{Cyl}(X)$ above can also be thought of as the smashout map $(1 \rightarrowtail I)\square j$. Thus j is a cofibration iff $(1 \rightarrowtail I)\square j$ is a split monomorphism.

The idea is that a cofibration is a homotopically well-behaved inclusion. The following theorem gives a convenient test.

Theorem 5.4. A map $i\colon X \to Y$ is a cofibration if and only if it is a closed inclusion (so we can think of X as a closed subspace of Y) and there are maps $u\colon Y \to I$ and $h\colon I \times Y \to Y$ such that

1. $u^{-1}\{0\} = X$
2. $h_1 = 1_Y$
3. $h_t|_X = 1_X$ for all t.
4. $h_0(y) \in X$ for all y such that $u(y) < 1$.

Here $h_t\colon Y \to Y$ is defined by $h_t(y) = h(t, y)$.

Moreover, i is an acyclic cofibration if and only if we can choose u, h such that $u(y) \leq 1/2$ for all y, and thus $h_1(Y) = X$.

Example 5.5.

1. A smooth closed embedding of manifolds is a cofibration.
2. The inclusion of a subcomplex in a simplicial complex is a cofibration.
3. The inclusion of a closed subvariety in a real or complex projective variety is a cofibration.
4. For an example in the other direction, consider the "Hawaian earrings" space E, which is the union of the circles of radius $1/n$ centred at $(1/n, 0) \in \mathbb{R}^2$ as n runs from 1 to ∞. The inclusion of the one-point space $\{(0, 0)\}$ in E is not a cofibration.

The following fact turns out to be crucial.

Proposition 5.6. The smashout of two cofibrations is a cofibration. If either one is a homotopy equivalence, then so is the smashout. The crossmap of a cofibration and a fibration is a fibration. If either one is a homotopy equivalence, then so is the crossmap.

The following *homotopy extension property* of cofibrations is very useful.

Proposition 5.7. Let X be a closed subspace of a space Y, such that the inclusion map $i\colon X \to Y$ is a cofibration. Suppose we are given a map $f\colon Y \to Z$, and a homotopy $g_t\colon X \to Z$ ending with $g_1 = f|_X$. Then the homotopy can be extended to give a homotopy $h_t\colon Y \to Z$ with $h_t|_X = g_t$ and $h_1 = f$.

Conversely, if X is any closed subspace of Y with this homotopy extension property, then the inclusion $X \to Y$ is a cofibration.

The dual concept is that of a fibration.

Definition 5.8. Let $q\colon E \to B$ be a map. Define

$$\mathrm{Path}(B) = C(I, B) = \{ \text{ continuous paths } \omega\colon I \to B\}.$$

Let $p_1\colon \mathrm{Path}(B) \to B$ be the map $p_1(\omega) = \omega(1)$, and define the *mapping path space* $\mathrm{Path}(q) = C(I, B) \times_B E$ by the following pullback:

$$\begin{tikzcd}
\mathrm{Path}(i) \arrow[r] \arrow[d] & \mathrm{Path}(B) \arrow[d, "p_1"] \\
E \arrow[r, "q"'] & B.
\end{tikzcd}$$

In other words,

$$\mathrm{Path}(q) = \{(\omega, e) \in C(I, B) \times E \mid \omega(1) = q(e)\},$$

so a point of $\mathrm{Path}(q)$ is a path in B together with a lift of the final point to E. There is an evident map $r\colon \mathrm{Path}(E) \to \mathrm{Path}(q)$, given by $r(\alpha) = (q \circ \alpha, \alpha(1))$. We say that q is a (Hurewicz) fibration iff there is a map $l\colon \mathrm{Path}(q) \to \mathrm{Path}(E)$ such that $rl = 1$. This means that $\alpha = l(\omega, e)$ is a path in E which is a lift of ω (in the sense that $q \circ \alpha = \omega$), ending at $\alpha(1) = e$, which is our given lift of $\omega(1)$. Such a map l is called a *path-lifting function* for q. A fibration which is also a homotopy equivalence is called an *acyclic fibration*. We write fib and afb for the classes of fibrations and acyclic fibrations.

Remark 5.9. The map $r\colon \mathrm{Path}(E) \to \mathrm{Path}(q)$ defined above can also be thought of as the crossmap $F(1 \rightarrowtail I, q)$. Thus, q is a fibration iff $F(1 \rightarrowtail I, q)$ is a split epimorphism.

The idea is that a fibration is a homotopically well-behaved projection map.

Example 5.10. Under mild technical conditions (for example, if the base space is a metric space or a CW complex — see Appendix B), every locally-trivial fibre bundle is a fibration. Here are some interesting examples of locally-trivial fibre bundles.

(1) Covering maps; in particular, quotients by discrete group actions; for example, the covering of a compact Riemann surface of genus greater than one by the unit disc.
(2) Maps of the configuration spaces $F_k\mathbb{C}$.
(3) The fibration $O(n-1) \to O(n) \xrightarrow{q} S^{n-1}$, where $O(n)$ is the space of $n \times n$ orthogonal matrices, and $q(A) = Ae_n$, where e_n is the last basis vector in $\mathbb{R}^n$. (When we say that $F \to E \xrightarrow{q} B$ is a fibration, we mean that q is a fibration and $F = \mathrm{fib}(q)$.)
(4) The fibration $\mathbb{C}P^{n-1} \to H \xrightarrow{q} \mathbb{C}P^m$, where $m \leq n$, H is Milnor's hypersurface $\{([z],[w]) \in \mathbb{C}P^n \times \mathbb{C}P^m \mid \sum_{i=0}^m z_i w_i = 0\}$, and $q([z],[w]) = [w]$.
(5) The fibration $S^1 \to S^{2n+1} \xrightarrow{q} \mathbb{C}P^n$. The case $n = 1$ is especially interesting. There we can identify S^3 with $\{(z,w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\}$ and $\mathbb{C}P^1$ with $\mathbb{C} \cup \{\infty\}$, and q with the map $(z,w) \mapsto z/w$. As $\mathbb{C}P^1$ is also homeomorphic to S^2, we get a fibration $S^1 \to S^3 \to S^2$.
(6) It can be shown that any surjective submersion of manifolds is a locally trivial bundle. (A smooth map $f\colon M \to N$ is said to be a *submersion* if the induced map of tangent spaces $T_xM \to T_{f(x)}N$ is surjective for all $x \in M$.)

Example 5.11. We now list some fibrations which are not locally trivial bundles.

(1) For any space X, there is a fibration $\Omega X \to PX \xrightarrow{p_1} X$.
(2) For any $n > 0$ there is a 2-local fibration $S^n \xrightarrow{\eta} \Omega S^{n+1} \xrightarrow{H} \Omega S^{2n+1}$, called the EHP fibration. This really means that there is a map $H\colon \Omega S^{n+1} \to \Omega S^{2n+1}$ and a map $f\colon S^n \to PH$ which induces an isomorphism $\pi_*(S^n)/2 \simeq \pi_*(PH)/2$, and that the composite $S^n \to PH \to \Omega S^{n+1}$ is η.
(3) It can be shown that the homotopy fibre of the inclusion $X \vee Y \to X \times Y$ is homotopy equivalent to $\Sigma((\Omega X) \wedge (\Omega Y))$.

Example 5.12. If $B = \mathbb{R}$ and

$$E = \{(x,y) \in \mathbb{R}^2 \mid (x \leq 0 \text{ and } |y| = |x|) \text{ or } (x \geq 0 \text{ and } y = 0)\}$$

then the vertical projection $E \to B$ is not a fibration.

Theorem 5.13.

(1) A map is a cofibration if and only if it is left orthogonal to all maps of the form $p_1\colon \mathrm{Path}(B) \twoheadrightarrow B$, iff it is left orthogonal to every acyclic fibration. In symbols:

$$\mathrm{cof} = {}^{\perp}\{\text{ maps of the form } p_1\colon \mathrm{Path}(B) \twoheadrightarrow B\} = {}^{\perp}\,\mathrm{afb}\,.$$

(2) A map is a fibration if and only if it is right orthogonal to all maps of the form $i_1\colon X \rightarrowtail \mathrm{Cyl}(X)$, iff it is right orthogonal to every acyclic cofibration. In symbols:

$$\mathrm{fib} = \{\text{ maps of the form } i_1\colon X \rightarrowtail \mathrm{Cyl}(X)\}^{\perp} = \mathrm{acf}^{\perp}\,.$$

We also have $\mathrm{acf} = {}^{\perp}\,\mathrm{fib}$ and $\mathrm{afb} = \mathrm{cof}^{\perp}$.

Corollary 5.14. The classes cof and acf are closed under composition, disjoint unions, pushouts, retractions, and sequential colimits. The classes fib and afb are closed under compositions, products, pullbacks, retractions, and sequential inverse limits. All four classes contain all homeomorphisms.

Proposition 5.15. Any map $f\colon X \to Y$ can be fitted into a natural commutative diagram as shown, in which the arrow marked cof is a cofibration and so on.

$$\begin{tikzcd}
X \arrow[r, tail, "\mathrm{cof}"] \arrow[d, "\mathrm{heq}"'] & \mathrm{Cyl}(f) \arrow[d, "\mathrm{heq}"] \\
\mathrm{Path}(f) \arrow[r, two heads, "\mathrm{fib}"'] & Y.
\end{tikzcd}$$

The following propositions are also useful.

Proposition 5.16. If X is compact and $j\colon Y \to Z$ is a cofibration then the map $j_*\colon C(X,Y) \to C(X,Z)$ is a cofibration.

Proposition 5.17. Suppose that we have a pullback square as follows, in which j is a cofibration and q is a fibration. Then j' is a cofibration and q' is a fibration.

$$\begin{tikzcd}
E' \arrow[r, "j'"] \arrow[d, "q'"'] & E \arrow[d, "q"] \\
B' \arrow[r, "j"'] & B.
\end{tikzcd}$$

6. Eilenberg-MacLane spaces

Definition 6.1. Let $n > 0$ be an integer, and A a group, which is Abelian if $n > 1$. A *space of type* $K(A,n)$ is a pointed space X such that $\pi_k X = 0$ for $k \neq n$, equipped with an isomorphism $u\colon \pi_n X \to A$. We also require that X has the homotopy type of a CW complex.

Theorem 6.2. For any n and A as above, there exists a space X of type $K(A,n)$. If X and Y are two such spaces, with given isomorphisms $\pi_n X \xrightarrow{u} A \xleftarrow{v} \pi_n Y$, then there is a unique homotopy class of maps $f\colon X \to Y$ such that $u = v \circ \pi_n(f)$, and any such map is a homotopy equivalence.

This theorem shows that there is an essentially unique space of type $K(A,n)$, so we can safely call any such space $K(A,n)$. In the case $n=1$, we also write BG for $K(G,1)$, and call it the classifying space of G.

Example 6.3.

1. $B\mathbb{Z} = K(\mathbb{Z},1) = S^1$.
2. $K(\mathbb{Z},2) = \mathbb{C}P^\infty$.
3. $BC_n = K(C_n,1) = S^\infty/C_n$.
4. $B\Sigma_n$ is the space of subsets of $\mathbb{R}^\infty$ of order n.
5. If M is any closed surface other than S^2 or $\mathbb{R}P^2$, then $M = BG$ for some group $G = \pi_1 M$. The same holds for most irreducible 3-manifolds.
6. If $\mathbb{Z}X$ is the free Abelian group generated by the points of X, and $\widetilde{\mathbb{Z}}X = \mathbb{Z}X/\mathbb{Z}\{0_X\}$ then $K(\mathbb{Z},n) = \widetilde{\mathbb{Z}}S^n$ for all $n > 0$.

Proposition 6.4. If A is Abelian then there is a natural homotopy equivalence $K(A,n) = \Omega K(A,n+1)$, and thus $K(A,n)$ is naturally a commutative H-group. Moreover, there is a natural isomorphism $[X, K(A,n)] = H^n(X;A)$ for any CW complex X.

Proposition 6.5. Any short exact sequence $A \to B \to C$ of groups gives a fibration $K(A,n) \to K(B,n) \to K(C,n)$ in a natural way.

Proposition 6.6. If a group G acts freely on a space X then under mild technical conditions there is a fibration up to homotopy $X \to X/G \to BG$.

Remark 6.7. Elements of $H^m(K(A,n);B)$ biject with homotopy classes of maps $f\colon K(A,n) \to K(B,m)$. Given such a map, we have a natural transformation

$$f_*\colon H^n(X;A) = [X, K(A,n)] \to [X, K(B,m)] = H^m(X;B),$$

defined by $f_*(u) = f \circ u$. These operations need not be additive or multiplicative. The groups $H^*(K(\mathbb{F}_p,n);\mathbb{F}_p)$ are known, so we have a good supply of natural operations in mod p cohomology. The additive operations form a ring called the Steenrod algebra, about which a great deal is known.

7. Groups and cogroups

Observe that a group is a pointed set G equipped with an addition map $\sigma\colon G \times G \to G$, and an inversion map $\nu\colon G \to G$ making the following diagrams commute:

$$\begin{tikzcd}
G \times G \times G \arrow[r, "\sigma\times 1"] \arrow[d, "1\times\sigma"'] & G \times G \arrow[r, no head, shift left, "0\times 1"] \arrow[r, no head, shift right, "1\times 0"'] \arrow[d, "\sigma"'] & G \arrow[dl, "1"] & G \arrow[r, "\Delta"] \arrow[dl] & G \times G \arrow[d, "1\times\nu"] \\
G \times G \arrow[r, "\sigma"'] & G & 0 \arrow[r, "0"'] & G & G \times G \arrow[l, "\sigma"]
\end{tikzcd}$$

Here $0\colon 0 \to G$ is the inclusion of the zero element. This motivates the following definition:

Definition 7.1. An *H-group* is a pointed space G equipped with maps $G \times G \xrightarrow{\sigma} G \xleftarrow{\nu} G$ making the above diagram commute up to homotopy.

We write things additively in the hope that this will persuade our H-groups to be commutative (at least up to homotopy); they cooperate more often than one might expect.

Of course, a topological group is an H-group. We give one slightly less obvious example.

Example 7.2. Let $G = \mathbb{C}P^\infty$ be the space of nonzero complex polynomials modulo multiplication by nonzero complex numbers (i.e. $\mathbb{C}P^\infty = (\mathbb{C}[t] \setminus \{0\})/\mathbb{C}^\times$). Let $\sigma\colon G^2 \to G$ be given by multiplication of polynomials, and let $0\colon 0 \to G$ be the map with value 1. Then 0 and σ make the left hand diagram above commute on the nose. There is no map $\nu\colon G \to G$ which makes the right hand diagram commute on the nose, but if we define

$$\nu[\sum_i a_i t^i] = [\sum_i \overline{a}_i t^i],$$

then the right hand diagram commutes up to homotopy.

Example 7.3. More generally, any space of type $K(A, n)$ (with A Abelian) is an H-group in a canonical way.

We also need the dual concept. For this we need to reverse all the arrows and change all the products to coproducts, which are given by the wedge.

Definition 7.4. An *H-cogroup* is a pointed space X equipped with maps $X \vee X \xleftarrow{\delta} X \xrightarrow{\zeta} X$ making the following diagrams commute:

$$\begin{tikzcd}
X \vee X \vee X & X \vee X \arrow[l, "\delta\vee 1"'] \arrow[r, shift left, no head, "0\vee 1"] \arrow[r, shift right, no head, "1\vee 0"'] & X & X & X \vee X \arrow[l, "\nabla"'] \\
X \vee X \arrow[u, "1\vee\delta"] & X \arrow[l, "\delta"] \arrow[u, "\delta"] \arrow[ur, "1"'] & 0 \arrow[ur] & X \arrow[l, "0"] \arrow[r, "\delta"'] & X \vee X \arrow[u, "1\vee\zeta"']
\end{tikzcd}$$

Example 7.5. We can make $S^1 = I/\{0, 1\}$ into an H-cogroup as follows. Define $\zeta[t] = [1 - t]$. Let i_0, i_1 be the two obvious inclusions $S^1 \rightarrowtail S^1 \vee S^1$. Define $\delta\colon S^1 \to S^1 \vee S^1$ by

$$\delta[t] = \begin{cases} i_0[2t] & \text{if } 0 \le t \le 1/2 \\ i_1[2t-1] & \text{if } 1/2 \le t \le 1. \end{cases}$$

It is not hard to check that this does the job. Similarly, for any pointed space X the suspension $\Sigma X = S^1 \wedge X$ is an H-cogroup with coproduct $\delta \wedge 1\colon \Sigma X \to (S^1 \vee S^1) \wedge X = \Sigma X \vee \Sigma X$. Dually, there is a natural identification $F(S^1 \vee S^1, X) = \Omega X \times \Omega X$, and this together with the map $\delta^*\colon F(S^1 \vee S^1, X) \to F(S^1, X)$ makes ΩX into an H-group.

We can modify the definition of ΩX to make it associative on the nose. To do this, write

$$\begin{aligned} P^*X &= \{(r, \alpha) \mid r \ge 0\ ,\ \alpha\colon [0, r] \to X\ ,\ \alpha(0) = 0_X\} \\ \Omega^*X &= \{(r, \alpha) \in P^*X \mid \alpha(r) = 0\}. \end{aligned}$$

These sets can be topologised in a natural way. We define a product on Ω^*X by

$$\mu((r, \alpha), (s, \beta)) = (r + s, \gamma),$$

where

$$\gamma(t) = \begin{cases} \alpha(t) & \text{if } 0 \le t \le r \\ \beta(t - r) & \text{if } r \le t \le r + s. \end{cases}$$

We can also define an inversion map by $\nu((r,\alpha)) = (r,\beta)$, where $\beta(t) = \alpha(r-t)$. It is not hard to check that $\Omega^* X$ becomes an H-group, and that the left hand diagram in the definition commutes on the nose, so that $\Omega^* X$ is actually a topological monoid.

Proposition 7.6. If X is a well-pointed space, then there is a homotopy equivalence $\Omega X \simeq \Omega^* X$.

This construction cannot be dualised, as shown by the following exercise:

Exercise 7.7. If X is an H-cogroup for which the left hand diagram commutes on the nose, then $X = \{0\}$.

Definition 7.8. Let $\tau = \tau_X \colon X \times X \to X \times X$ be the map $\tau(x,y) = (y,x)$. We also write τ for the restriction of this to $X \vee X = X \times \{0_X\} \cup \{0_X\} \times X \subseteq X \times X$. We say that an H-group G is *homotopy-commutative* if we have $\mu \simeq \mu \circ \tau$. Dually, we say that an H-cogroup X is *homotopy-cocommutative* if we have $\tau \circ \delta \simeq \delta$.

It turns out that many H-(co)groups are homotopy-(co)commutative. This can often be deduced from the following lemma:

Lemma 7.9. Let X be a set with two binary operations $*, \circ$ with the same unit element 0, such that $(a * b) \circ (c * d) = (a \circ c) * (b \circ d)$ for all $a, b, c, d \in X$. Then $* = \circ$ and this operation is commutative.

In particular, one can deduce the following.

Proposition 7.10. If X is an H-cogroup then the two natural H-cogroup structures on ΣX are the same and are cocommutative. If G is an H-group then the two natural H-group structures on ΩX are the same and are commutative.

Corollary 7.11. If $n > 1$ then $\Sigma^n X$ has a natural structure as a cocommutative H-cogroup, and $\Omega^n X$ has a natural structure as a commutative H-group.

8. Fibres and cofibres

Let $f \colon X \to Y$ be a map of well-pointed spaces. We define the geometric fibre $\mathrm{fib}(f)$ and cofibre $\mathrm{cof}(f)$ by

$$\mathrm{fib}(f) = f^{-1}\{0_Y\} = \{x \in X \mid f(x) = 0_Y\}$$
$$\mathrm{cof}(f) = Y/\overline{f(X)}.$$

Note that we have a commutative diagram as follows, in which the left square is a pullback and the right square is a pushout:

$$\begin{tikzcd}
\mathrm{fib}(f) \arrow[r] \arrow[d] & X \arrow[r, "i_1"] \arrow[d, "f"] & 0 \arrow[d] \\
0 \arrow[r, "p_1"'] & Y \arrow[r] & \mathrm{cof}(f).
\end{tikzcd}$$

The construction of $\mathrm{cof}(f)$ will not be homotopically well-behaved unless f is a cofibration, and similarly $\mathrm{fib}(f)$ is only a reasonable definition when f is a fibration. We next define modified versions which are always well-behaved.

We define the homotopy fibre Pf and cofibre Cf so that the left square below is a pullback, and the right square is a pushout. This is the same as the previous diagram except that we have replaced the two copies of 0 by contractible spaces CX and PY.

$$\begin{tikzcd}
Pf \arrow[r, "p"] \arrow[d] & X \arrow[r, "i_1"] \arrow[d, "f"] & CX \arrow[d] \\
PY \arrow[r, "p_1"'] & Y \arrow[r, "i"'] & Cf.
\end{tikzcd}$$

More explicitly,

$$Pf = \{(x,\omega) \mid \omega\colon I \to Y\ ,\ \omega(0) = 0\ ,\ \omega(1) = f(x)\}$$
$$Cf = I \times X \amalg Y/\sim,$$

where the equivalence relation $\sim$ collapses $\{0\} \times X$ and $I \times \{0_X\}$ to a point and identifies $(1,x) \in I \times X$ with $f(x) \in Y$. Thus in Cf we adjoin a path from $f(x)$ to the basepoint, rather than actually identifying it with the basepoint. In the definition of Pf, we do not require that $f(x) = 0_Y$, but rather that we are given a path ω from 0_Y to $f(x)$.

Remark 8.1. There is a natural map $e\colon \Omega Y \to Pf$, sending ω to $(0_X,\omega)$. Dually, there is a natural map $d\colon Cf \to \Sigma X$, sending $Y \subseteq Cf$ to the basepoint, and $I \times X$ to ΣX by the usual projection.

In a reasonably strong sense, these constructions only depend on the homotopy class of f.

Proposition 8.2. A homotopy $h\colon f \simeq f'$ gives rise to a diagram as follows:

$$\begin{tikzcd}
\Omega Y \arrow[r, "e"] \arrow[d, equal] & Pf \arrow[r, "p"] \arrow[d, "\alpha(h)"', "\simeq"] & X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "i"] \arrow[d, equal] & Cf \arrow[r, "d"] \arrow[d, "\simeq"', "\beta(h)"] & \Sigma X \arrow[d, equal] \\
\Omega Y \arrow[r, "e'"'] & Pf' \arrow[r, "p'"'] & X \arrow[r, "f"'] & Y \arrow[r, "i'"'] & Cf' \arrow[r, "d'"'] & \Sigma X
\end{tikzcd}$$

The middle three squares commute on the nose, and the outer two commute up to homotopy. The maps $\alpha(h)$ and $\beta(h)$ are homotopy equivalences.

In some sense both Pf and Cf measure the failure of f to be a homotopy equivalence. By analogy with algebra, you might expect them to give different kinds of information; in some sense, Pf might measure the failure of injectivity, and Cf the lack of surjectivity. However, it turns out that both kinds of information are mixed up together and are present in both Cf and Pf. The following proposition relates the information given by Cf and Pf.

Proposition 8.3. There is a natural map $g\colon \Sigma Pf \to Cf$, and an adjoint map $g^{\#}\colon Pf \to \Omega Cf$.

In Theorem 10.1 we will show that g is compatible with various other natural maps. In Theorem 14.2 we will give conditions that ensure that g is a homotopy equivalence in low dimensions.

Proposition 8.4. There are natural maps $u\colon Cf \to \mathrm{cof}(f)$ and $v\colon \mathrm{fib}(f) \to Pf$ such that ui is the obvious quotient map $Y \twoheadrightarrow \mathrm{cof}(f)$ and qv is the inclusion $\mathrm{fib}(f) \rightarrowtail X$. If f is a cofibration then u is a homotopy equivalence, and if f is a fibration then v is a homotopy equivalence.

Proposition 8.5. There is a natural isomorphism $\widetilde{H}_kX = \widetilde{H}_{k+1}\Sigma X$, and similarly in cohomology.

If $f\colon X \to Y$ is an inclusion map, then $H_*(Y,X) = \widetilde{H}_*Cf$. Even when f is not an inclusion, there is a long exact sequence

$$\ldots \leftarrow H_{k-1}X \leftarrow \widetilde{H}_kCf \leftarrow H_kY \leftarrow H_kX \leftarrow \widetilde{H}_{k+1}(Y,X) \leftarrow \ldots.$$

There is a similar long exact sequence in cohomology, with the arrows reversed.

The following weakenings of the notions of homotopy equivalences and fibrations are sometimes useful.

Definition 8.6. We say that a map $f\colon X \to Y$ is a *weak equivalence* if the induced map $\pi_*(X) \to \pi_*(Y)$ is an isomorphism, for any choice of basepoint in X.

Definition 8.7. Let $q\colon E \to B$ be a map of (not necessarily pointed) spaces. Given any point $b \in B$, we define

$$P_bq = \{(e,\omega) \in E \times \mathrm{Path}(B) \mid \omega(0) = b\ ,\ \omega(1) = q(e)\}.$$

(Thus if we choose a point $e \in E$ with $qe = b$ and give E and B the basepoints e and b, we get $Pq = P_bq$.) There is a natural map $q^{-1}\{b\} \to P_bq$. If this map is a weak equivalence for all $b \in B$, we say that q is a *quasifibration*.

Example 8.8.

1. Write $E = [-1,0] \times \{0\} \cup [0,1] \times [-1,1]$ and $B = [-1,1]$. Then the vertical projection $q\colon E \to B$ is a quasifibration but not a fibration.
2. Let $\widetilde{\mathbb{Z}}X$ be the free Abelian group generated by the points of X, modulo the subgroup generated by the basepoint. If $f\colon X \to Y$ is a cofibration, it can be shown that $\widetilde{\mathbb{Z}}X \to \widetilde{\mathbb{Z}}Y \to \widetilde{\mathbb{Z}}(Y/X)$ is a quasifibration.

9. Actions and coactions

Definition 9.1. We say that a group G acts on a set Z if there is a given map $\alpha\colon G \times Z \to Z$ such that $\alpha \circ (0 \times 1_Z) = 1_Z$ and

$$\alpha \circ (\mu \times 1_Z) = \alpha \circ (1_G \times \alpha)\colon G \times G \times Z \to Z.$$

Similarly, we say that an H-group G acts on a space Z if there is a given map $\alpha\colon G \times Z \to Z$ such that the above relations hold up to homotopy. Dually, we say that an H-cogroup X coacts on Z if there is a given map $\beta\colon Z \to X \vee Z$ such that $(0 \vee 1_Z) \circ \beta \simeq 1_Z$ and

$$(\delta \vee 1_Z) \circ \beta \simeq (1_X \vee \beta) \circ \beta\colon Z \to X \vee X \vee Z.$$

Remark 9.2. If $f\colon G \to H$ is a map of H-groups (by which we mean a homomorphism up to homotopy, in the evident sense) then there is a natural action of G on H given by the composite

$$G \times H \xrightarrow{f\times 1} H \times H \xrightarrow{\mu_H} H.$$

Dually, if $f\colon X \to Y$ is a map of H-cogroups then there is a natural coaction of Y on X, given by the composite

$$X \xrightarrow{\delta_X} X \vee X \xrightarrow{f\vee 1} Y \vee X.$$

Proposition 9.3. Let $f\colon X\to Y$ be a map of well-pointed spaces. Then ΩY acts naturally on itself, on Pf, and on ΩCf, and the maps

$$\Omega Y \xrightarrow{e} Pf \xrightarrow{g} \Omega Cf$$

are compatible with these actions. Dually, there are natural coactions of ΣX on Y, Cf and ΣPf, compatible with the maps

$$\Sigma Pf \xrightarrow{g^{\#}} Cf \xrightarrow{d} \Sigma X.$$

Proposition 9.4. There are diagrams as follows, in which the left square is a homotopy pushout and the right square is a homotopy pullback.

$$\begin{tikzcd}
Y \arrow[r, "i"] \arrow[d, "i"'] & Cf \arrow[d, "\text{incl}"] \\
Cf \arrow[r, "\text{coact}"'] & \Sigma X \vee Cf
\end{tikzcd}
\qquad
\begin{tikzcd}
\Omega Y \times Pf \arrow[r, "\text{act}"] \arrow[d, "\text{proj}"'] & Pf \arrow[d, "p"] \\
Pf \arrow[r, "p"'] & X.
\end{tikzcd}$$

10. Puppe sequences

We write $\zeta\colon S^1\to S^1$ for the map $\zeta[t]=[1-t]$. We also consider the counit and unit maps

$$\Sigma\Omega X \xrightarrow{\epsilon} X \xrightarrow{\eta} \Omega\Sigma X$$

defined by

$$\begin{aligned}
\epsilon(t\wedge\omega) &= \omega(t)\\
\eta(x)(t) &= t\wedge x.
\end{aligned}$$

Theorem 10.1. There is a commutative diagram as follows.

$$\begin{tikzcd}
 & & \Sigma\Omega X \arrow[r, "\Sigma\Omega f"] \arrow[d, "\epsilon"'] & \Sigma\Omega Y \arrow[r, "\Sigma e"] \arrow[d, "\epsilon"'] & \Sigma Pf \arrow[r, "\Sigma p"] \arrow[d, "g^{\#}"] & \Sigma X \arrow[d, "\simeq"', "\zeta\wedge 1"] \\
\Omega Y \arrow[r, "e"] \arrow[d, "\zeta^*"', "\simeq"] & Pf \arrow[r, "p"] \arrow[d, "g"'] & X \arrow[r, "f"] \arrow[d, "\eta"'] & Y \arrow[r, "i"'] \arrow[d, "\eta"] & Cf \arrow[r, "d"'] & \Sigma X \\
\Omega Y \arrow[r, "\Omega i"'] & \Omega Cf \arrow[r, "\Omega d"'] & \Omega\Sigma X \arrow[r, "\Omega\Sigma f"'] & \Omega\Sigma Y & &
\end{tikzcd}$$

We temporarily write i_f for the map $i\colon Y\to Cf$, to emphasise the dependence on f. Having constructed the sequence $X\xrightarrow{f}Y\xrightarrow{i_f}Cf$, it is natural to go one step further and consider the map $i_{i_f}\colon Cf\to Ci_f$. However, it turns out that Ci_f is homotopy equivalent to ΣX. More precisely, we have the following result:

Proposition 10.2. There is a natural map $b_f\colon Ci_f \to \Sigma X$ which is a homotopy equivalence, and which makes the following diagram commute.

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \arrow[r, "i_f"] & Cf \arrow[r, "i_{i_f}"] \arrow[dr, "d_f"'] & Ci_f \arrow[r] \arrow[d, "b_f"] \arrow[dr, "d_{i_f}"] & Ci_{i_f} \arrow[d, "b_{i_f}"] \\
& & & \Sigma X \arrow[r, "\zeta\wedge f"'] & \Sigma Y.
\end{tikzcd}$$

Dually, there is a natural homotopy equivalence $c_f\colon \Omega Y \to Pp_f$ making the following diagram commute.

$$\begin{tikzcd}
Pp_{p_f} \arrow[r] & Pp_f \arrow[r, "p_{p_f}"] & Pf \arrow[r, "p_f"] & X \arrow[r, "f"] & Y \\
\Omega X \arrow[u, "c_{p_f}"] \arrow[ur, "e_{p_f}"'] \arrow[r, "\zeta^*\circ\Omega f"'] & \Omega Y \arrow[u, "c_f"'] \arrow[ur, "e_f"'] & & &
\end{tikzcd}$$

Definition 10.3. A sequence of pointed sets

$$A_0 \xleftarrow{f_0} A_1 \xleftarrow{f_1} A_2 \xleftarrow{f_2} A_3 \xleftarrow{f_3} \dots$$

is an *unstable long exact sequence (ULES)* if

1. $f_{i-1}^{-1}\{0\} = f_i(A_{i+1}) \subseteq A_i$ for all $i \geq 1$.
2. A_i is a group for $i \geq 3$, which is Abelian if $i \geq 6$.
3. A_3 acts on A_2, and f_2 is compatible with the actions of A_3 on A_3 and A_2.
4. f_1 induces an injection from the orbit space A_2/A_3 to A_1.

Theorem 10.4. By applying the functor $[Z,-]$ to the sequence

$$Y \xleftarrow{f} X \xleftarrow{p} Pf \xleftarrow{e} \Omega Y \xleftarrow{\Omega f} \Omega X \xleftarrow{\Omega Pf} \dots,$$

we obtain a ULES

$$[Z,Y] \leftarrow [Z,X] \leftarrow [Z,Pf] \leftarrow [\Sigma Z, Y] \leftarrow \dots.$$

In particular, taking $Z = S^0 = \{0,1\}$ gives a ULES

$$\pi_0 Y \leftarrow \pi_0 X \leftarrow \pi_1(Y,X) \leftarrow \pi_1 Y \leftarrow \pi_1 X \leftarrow \dots.$$

Dually, by applying the functor $[-,Z]$ to the sequence

$$X \xrightarrow{f} Y \xrightarrow{i} Cf \xrightarrow{d} \Sigma X \xrightarrow{\Sigma f} \Sigma Y \xrightarrow{\Sigma i} \Sigma Cf \xrightarrow{d} \dots,$$

we obtain a ULES

$$[X,Z] \leftarrow [Y,Z] \leftarrow [Cf,Z] \leftarrow [X,\Omega Z] \leftarrow [Y,\Omega Z] \leftarrow [Cf,\Omega Z] \leftarrow \dots.$$

Definition 10.5. Given a pointed subspace $X \subseteq Y$ and an integer $n > 0$, we write $\pi_n(Y,X) = \pi_{n-1}(Pi)$, where $i\colon X \to Y$ is the inclusion.

Corollary 10.6. For any pointed subspace $X \subseteq Y$, there is a ULES of homotopy groups

$$\pi_0 Y \leftarrow \pi_0 X \leftarrow \pi_1(Y,X) \leftarrow \pi_1 Y \leftarrow \pi_1 X \leftarrow \pi_2(Y,X) \leftarrow \dots.$$

Example 10.7. LES for Hopf fibration, EHP fibrations, $U(n-1) \to U(n) \to S^{2n-1}$.

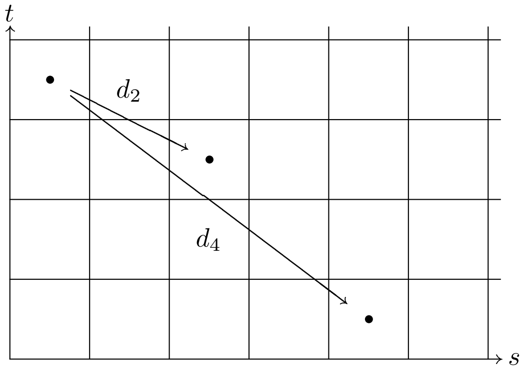
11. SPECTRAL SEQUENCES

Definition 11.1. A (multiplicative, first quadrant, cohomologically graded) spectral sequence consists of a sequence of "pages" E_r for $r \geq 2$. Each page is a bigraded Abelian group E_r^{st}, with $E_r^{st} = 0$ if $s < 0$ or $t < 0$. It come equipped with a differential

$$d_r \colon E_r^{st} \to E_r^{s+r,t-r+1}$$

satisfying $d_r^2 = 0$, and the next page E_{r+1} is the cohomology of E_r with respect to d_r:

$$E_{r+1}^{st} = \frac{\ker(d_r \colon E_r^{st} \to E_r^{s+r,t-r+1})}{\text{image}(d_r \colon E_r^{s-r,t+r-1} \to E_r^{st})}.$$



Moreover, there are product maps $E_r^{st} \otimes E_r^{uv} \to E_r^{s+u,t+v}$ making E_r into a bigraded ring. It is commutative up to sign, and d_r is a derivation: if $a \in E_r^{st}$ and $b \in E_r^{uv}$ then

$$ab = (-1)^{(s+t)(u+v)} ba$$
$$d_r(ab) = d_r(a)b + (-1)^{s+t} a d_r(b).$$

Note that for fixed s and t, when r is sufficiently large the differential d_r starting at E_r^{st} ends below the s axis, and thus is zero; and the differential d_r ending at E_r^{st} starts to the left of the t axis, and thus is also zero. It follows that $E_r^{st} = E_{r+1}^{st}$ when $r \gg 0$. We write E_∞^{st} for this group.

We say that the spectral sequence *converges* to a graded ring A^* if there is a given filtration

$$A^u = F^0A^u \geq F^1A^u \geq \ldots \geq F^uA^u \geq F^{u+1}A^u = 0$$

such that $F^sA^u.F^tA^v \subseteq F^{s+t}A^{u+v}$, and given isomorphisms

$$F^sA^u/F^{s+1}A^u = E_\infty^{s,u-s}$$

that are compatible with the ring structures on E_∞ and A.

If so, note that there are *edge maps*

$$A^u \twoheadrightarrow F^0A^u/F^1A^u = E_\infty^{0,u} \rightarrowtail E_2^{0,u}$$
$$E_2^{u,0} \twoheadrightarrow E_\infty^{u,0} = F^uA^u \rightarrowtail A^u.$$

Remark 11.2. In these notes, we only consider the Serre spectral sequence, but there are many other spectral sequences for computing homotopy and homology groups in other circumstances. They can be used both for specific calculations (see the examples below) and for more theoretical arguments (see Section 13).

The main theorem is that for any fibration $F \to E \to B$, there are spectral sequences relating the (co)homology of F, E and B. We first give a theorem in which we make some restrictive assumptions to simplify the statement.

Theorem 11.3. Let $F \to E \xrightarrow{q} B$ be a fibration, with B simply connected. Let K be a field, take all cohomology with coefficients in K, and assume that H^nB and H^nF are finite-dimensional for all n. Then there is a *Serre spectral sequence* with $E_2^{st} = H^s(B) \otimes_K H^t(F)$, which converges to the ring H^*E. (The last sentence is often written: there is a Serre spectral sequence $H^s(B) \otimes_K H^t(F) \Longrightarrow H^{s+t}E$.)

We now give a more complicated statement which is more generally valid.

Theorem 11.4. Let $q\colon E \to B$ be a fibration, and R a commutative ring. Then there is a Serre spectral sequence

$$H^s(B; \mathcal{H}^t(F; R)) \Longrightarrow H^{s+t}(E; R),$$

where $\mathcal{H}^t(F; R)$ means the local coefficient system $b \mapsto H^t(q^{-1}\{b\}; R)$. Similarly, if B' is a subspace of B and $E' = q^{-1}B'$ then there is a relative Serre spectral sequence

$$H^s(B, B'; \mathcal{H}^t(F; R)) \Longrightarrow H^{s+t}(E; R).$$

(This does not have a ring structure, but it does have a module structure over the previous spectral sequence.)

We will say nothing about the theory of local coefficient systems except to explain when they are unnecessary. Recall that there is a natural action of the H-group ΩB on the fibre $Pq \simeq F$. Using this, each element of $\pi_0\Omega B = \pi_1 B$ gives a homotopy class of maps $F \to F$, and thus a map $H^*(F; R) \to H^*(F; R)$. This construction gives an action of the group $\pi_1 B$ on H^*F.

Proposition 11.5. If $F \to E \xrightarrow{q} B$ is a fibration, B is connected, and $\pi_1 B$ acts trivially on $H^*(F; R)$, then the E_2 terms of the above spectral sequences are just $H^s(B; H^t(F; R))$ and $H^s(B, B'; H^t(F; R))$.

Proposition 11.6. If E and B are H-groups, and $q\colon E \to B$ is both a fibration and an H-map, then $\pi_1 B$ acts trivially on F.

We also have Serre spectral sequences in homology (as opposed to cohomology). We give another definition to summarise their properties.

Definition 11.7. A first quadrant homologically graded spectral sequence consists of pages E^r for $r \geq 2$, with $E^r_{st} = 0$ if $s < 0$ or $t < 0$. There are differentials $d^r\colon E^r_{st} \to E^r_{s+r,t-r+1}$ (the opposite direction to the cohomological case) with $(d^r)^2 = 0$ and $E^{r+1} = \ker(d^r)/\operatorname{image}(d^r)$. We say that such a spectral sequence converges to a graded group A_* if there is a filtration $0 = F_{-1}A_u \leq F_0A_u \leq \ldots \leq F_uA_u = A_u$ and isomorphisms $F_sA_u/F_{s-1}A_u \simeq E^\infty_{s,u-s}$.

Theorem 11.8. If $q\colon E \to B$ is a fibration and R is a ring then there is a homologically graded Serre spectral sequence

$$H_s(B; \mathcal{H}_t(F; R)) \Longrightarrow H_{s+t}(E; R).$$

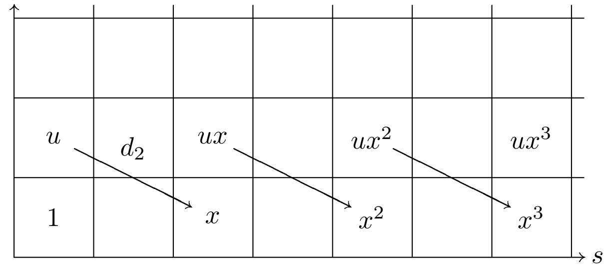
There is also a relative version.

12. Examples of the Serre spectral sequence

Example 12.1. Consider the fibration $S^1 \to S^{2n+1} \to \mathbb{C}P^n$. There is a Serre spectral sequence $H^*(\mathbb{C}P^n) \otimes H^*(S^1) \Longrightarrow H^*S^{2n+1}$. Recall that

$$\begin{aligned} H^*S^1 &= \mathbb{Z}[u]/u^2 & |u| &= 1 \\ H^*S^{2n+1} &= \mathbb{Z}[v]/v^2 & |v| &= 2n+1 \\ H^*\mathbb{C}P^n &= \mathbb{Z}[x]/x^{n+1} & |x| &= 2. \end{aligned}$$

Thus $E_2 = \mathbb{Z}[u,x]/(u^2, x^{n+1})$ with $u \in E_2^{01}$ and $x \in E_2^{20}$. It turns out that $d_2(u) = x$ and $d_2(x) = 0$, and thus that $d_2(x^k u) = x^{k+1}$ and $d_2(x^k) = 0$. It follows that $E_3 = \mathbb{Z}\{1, ux^n\}$, and there are no more differentials, so $E_3 = E_\infty$. The filtration on $H^{2n+1}S^{2n+1}$ is given by $F^0 = \ldots = F^{2n} = \mathbb{Z}v$, and $F^{2n+1} = 0$, and the isomorphism $F^{2n}/F^{2n+1} = E_\infty^{2n,1}$ sends v to $\pm ux^n$. We illustrate the case $n = 3$:



Example 12.2. Given integers $k \leq l$, we can define the Milnor hypersurface

$$M = \{([z],[w]) \in \mathbb{C}P^k \times \mathbb{C}P^l \mid \sum_{i=0}^{k} z_i w_i = 0\}.$$

Recall that

$$\begin{aligned} H^*\mathbb{C}P^l &= \mathbb{Z}[x]/x^{l+1} \\ H^*\mathbb{C}P^k &= \mathbb{Z}[y]/y^{k+1} \end{aligned}$$

There are obvious maps $p \colon M \to \mathbb{C}P^k$ and $q \colon M \to \mathbb{C}P^l$, defined by $p([z],[w]) = [z]$ and $q([z],[w]) = [w]$. We write x for $p^*x \in H^2M$ and y for $q^*y \in H^2M$. It turns out that

$$H^*M = \mathbb{Z}[x,y]/(x^{k+1}, y^l - y^{l-1}x + \ldots \pm x^l).$$

This has a basis $H^*M = \mathbb{Z}\{x^i y^j \mid i \leq k \ , \ j < l\}$. One can show that there is a fibration $\mathbb{C}P^{l-1} \to M \xrightarrow{p} \mathbb{C}P^k$. The natural filtration of H^*M arising from this filtration is given by

$$F^{2s}H^*M = F^{2s-1}H^*M = \text{ ideal generated by } y^s.$$

The associated graded ring $G^*H^*M = \prod_s F^s/F^{s+1}$ is given by

$$G^*H^*M = \mathbb{Z}[\overline{x}, \overline{y}]/(\overline{x}^{k+1}, \overline{y}^l),$$

where $\overline{x} \in G^2$ is the image of x in F^2/F^3, and $\overline{y} \in G^0$ is the image of y in F^0/F^1. (Because $y^l = y^{l-1}x - \ldots \mp x^l \in F^1$, we have $\overline{y}^l = 0$ in the natural ring structure on G^*.) Thus, G^* has rather simpler structure than H^*M does.

There is a Serre spectral sequence $H^*(\mathbb{C}P^k)\otimes H^*(\mathbb{C}P^l) \Longrightarrow H^*M$. The E_2 page is just $\mathbb{Z}[x,z]/(x^{k+1},z^l)$. Note that x and z both have even total degree (where the *total degree* of E_r^{st} is $s+t$) and that all differentials run from a slot of even total degree to one of odd total degree or *vice versa*, so all differentials are necessarily zero. It follows that $E_\infty = E_2$, and we have an obvious identification of E_∞ with G^*.

If we were using this spectral sequence to calculate H^*M, then we might be misled into believing that $y^l = 0$ in H^*M. This example shows that care is needed in deducing the multiplicative structure of the target ring H^*E from the E_∞ page of the spectral sequence.

We illustrate the case $k = l = 3$. We have written in the elements $d_r y^2$ to show that they all lie in slots where either s or t is odd, so they must all be zero.

t

y^2		xy^2		x^2y^2		x^3y^2
		d_2y^2				
y		xy	d_3y^2	x^2y		x^3y
				d_4y^2		
1		x		x^2	d_5y^2	x^3

s

Example 12.3. Consider the fibration $U(n-1) \xrightarrow{j} U(n) \xrightarrow{q} S^{2n-1}$, where $q(A) = Ae_n$. We will use this to calculate the cohomology of $U(n)$. We claim that there are canonically defined elements $a_{2k+1} \in H^{2k+1}U(n)$ for $0 \le k < n$ such that

$$H^*U(n) = E[a_1, a_3, \dots, a_{2n-1}].$$

To see this, assume the corresponding thing for $U(n-1)$ and consider the Serre spectral sequence $H^*S^{2n-1}\otimes H^*U(n-1) \Longrightarrow H^*U(n)$. Let u be the generator of $H^{2n-1}S^{2n-1}$, so the E_2 page is $E[a_{2i+1} \mid 0 \le i < n-1]\otimes E[u]$, with $a_{2i+1} \in E_2^{0,2i+1}$ and $u \in E_2^{2n-1,0}$. The whole page is thus concentrated in the 0'th and $(2n-1)$'st columns, and the only possible differential is d_{2n-1}. For each $i < n-1$ we have $d_{2n-1}(a_{2i-1}) \in E_{2n-1}^{2n-1,2(i-n)}$, which lies below the axis $t = 0$ and thus is zero. It is even easier to see that $d_{dn-1}(b) = 0$. As d_{2n-1} is a derivation, we see that it vanishes on the whole algebra generated by the a_i's and b, which is the whole E_{2n-1} page, so $E_\infty = E_2$. This means that in the natural filtration of $H^*U(n)$, the quotient $F^0/F^1 = E_\infty^{0*}$ maps isomorphically by j^* to $H^*U(n-1)$, that $F^1 = F^2 = \dots = F^{2n-1}$, that $F^{2n} = 0$, and that F^{2n-1} is a free module over F^0/F^1 on one generator q^*u.

For each $i < n-1$, the group $E_\infty^{0,2i+1}$ is the only nonzero term in total degree $2i+1$. It follows easily that there is a unique element $b_{2i+1} \in H^{2i+1}U(n)$ with $j^*b_{2i+1} = a_{2i+1}$. We also define $b_{2n-1} = q^*u$. All the b's lie in odd degrees, so they anticommute. We thus get a map $E[b_1, \dots, b_{2n-1}] \to H^*U(n)$. The element

b_{2n-1} lies in F^1, so we get a map $E[b_1, \dots, b_{2n-3}] \to F^0/F^1$. Given that j^* induces an isomorphism $F^0/F^1 \to E[a_1, \dots, a_{2n-3}]$ and $j^*b_{2i-1} = a_{2i-1}$, we conclude that our map $E[b_1, \dots, b_{2n-3}] \to F^0/F^1$ is an isomorphism. As F^{2n-1} is a free module over F^0/F^1 on one generator b_{2n-1}, we conclude that our map $E[b_1, \dots, b_{2n-1}] \to H^*U(n)$ is also an isomorphism.

We illustrate the case $n = 3$.

t

a_1a_3					a_1a_3u
a_3					a_3u
		d_2a_3			
a_1			d_3a_3		a_1u
1				d_4a_3	1

s

13. The Hurewicz theorem

Proposition 13.1. Given a pointed space Y, there is a natural map $h\colon \pi_n(Y) \to H_n(Y)$, which is a group homomorphism when $n > 0$. This is called the *(absolute) Hurewicz map*. Given a pointed subspace $X \subseteq Y$, there is a natural map $h\colon \pi_n(Y,X) \to H_n(Y,X)$, called the *relative Hurewicz map*, which is a homomorphism when $n > 1$. These maps fit into a commutative ladder

$$\begin{tikzcd}
\pi_{k-1}X \arrow[d, "h"'] & \pi_k(Y,X) \arrow[l] \arrow[d, "h"'] & \pi_kY \arrow[l] \arrow[d, "h"'] & \pi_kX \arrow[l] \arrow[d, "h"'] & \cdots \arrow[l] \\
H_{k-1}X & H_k(Y,X) \arrow[l] & H_kY \arrow[l] & H_kX \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

Definition 13.2. We say that a space X is *n-connected* if $\pi_kX = 0$ for $k \leq n$; this is independent of the choice of basepoint. A map $f\colon X \to Y$ of 0-connected spaces is n-connected if $\pi_k(f)$ is an isomorphism for $k < n$ and an epimorphism for $k = n$, or equivalently if the space Pf is $(n-1)$-connected. We say that a pair $X \subseteq Y$ of 0-connected spaces is n-connected if the inclusion map $X \to Y$ is n-connected.

Remark 13.3. For many purposes it is convenient to talk about $(n-1)$-connected spaces (which "start in dimension n") rather than n-connected spaces.

Definition 13.4. For any group G, we write $[G,G]$ for the (normal) subgroup generated by all commutators $[a,b] = aba^{-1}b^{-1}$, and G_{ab} for the Abelianisation $G/[G,G]$. Note that any map from G to an Abelian group factors uniquely through G_{ab}.

Theorem 13.5.

(1) If X is 0-connected, then h induces an isomorphism $\pi_1(X)_{\text{ab}} = H_1X$.

(2) If $n > 1$ then X is $(n-1)$-connected iff $\pi_1(X) = 0$ and $H_kX = 0$ for $k < n$. If so, then h gives an isomorphism $\pi_nX = H_nX$.

(3) If $n > 1$ and X is 1-connected, then the pair (Y, X) is $(n-1)$-connected iff it is 1-connected and $H_k(Y, X) = 0$ for $k < n$. If so, then h gives an isomorphism $\pi_n(Y, X) = H_n(Y, X)$.

14. The Freudenthal theorem

Theorem 14.1. If $n > 0$ and X is $(n-1)$-connected then the unit map $\eta\colon X \to \Omega\Sigma X$ and the counit map $\epsilon\colon \Sigma\Omega X \to X$ are both $(2n-1)$-connected.

Theorem 14.2. If $2 \leq m \leq n$ and X and Y are $(m-1)$-connected and $f\colon X \to Y$ is $(n-1)$-connected, then the map $g\colon \Sigma Pf \to Cf$ of Proposition 8.3 is $(n+m-1)$-connected.

Corollary 14.3. For any space X and any $n > 0$, the map $\Omega^n\Sigma^nX \to \Omega^{n+1}\Sigma^{n+1}X$ is $(n-1)$-connected.

15. CW complexes

Definition 15.1. We say that a space Y is obtained from a subspace X by attaching n-cells if there is a family of unbased maps $u_i\colon B^n \to X$ (indexed by some set I) such that $u_i(S^{n-1}) \subseteq Y$, and the resulting diagram

$$\begin{tikzcd}
\coprod_I S^{n-1} \arrow[r, "u"] \arrow[d, tail] & X \arrow[d, tail] \\
\coprod_I B^n \arrow[r, "u"'] & Y
\end{tikzcd}$$

is a pushout. If so, note that $Y = X \amalg \coprod_I \overset{\circ}{B}{}^n$ as sets, that Y is the mapping cone of the evident based map $\bigvee_I S^{n-1}_+ = (\coprod_I S^{n-1})_+ \to X$, and that $X/Y = \bigvee_I S^n$. It is easy to see that $H_*(Y, X)$ is canonically isomorphic to the free Abelian group $\mathbb{Z}\{I\}$, concentrated in degree n. It can also be shown that if $n \geq 3$ and X is 0-connected, then $\pi_n(Y, X) = \mathbb{Z}[\pi_1X]\{I\}$, a free module over the group-ring $\mathbb{Z}[\pi_1X]$.

Definition 15.2. A *CW complex* is a space X with a given filtration $X_0 \subseteq X_1 \subseteq \ldots \subseteq X$ by closed subspaces such that

(1) X_0 is discrete.

(2) For each $n > 0$, X_n is obtained from X_{n-1} by attaching n-cells.

(3) $X = \bigcup_n X_n$.

(4) The topology on X is such that a subset $Y \subseteq X$ is closed iff $Y \cap X_n$ is closed in X_n for all n.

Note that X is (as a set) the disjoint union of the sets $X_n \setminus X_{n-1}$, which is in turn a disjoint union of spaces homeomorphic to $\overset{\circ}{B}{}^n$. These spaces are called the *open cells* of X (although they are not open as subspaces of X in general). We say that a CW complex is *finite* if it has only finitely many cells. A *subcomplex* of X is a closed subset which is the union of some set of open cells. It can be shown that any subcomplex is itself a CW complex in a natural way, and that the inclusion of a subcomplex is a cofibration.

Example 15.3.

1. $X = \mathbb{R}$ can be made into a CW complex, with $X_0 = \mathbb{Z}$ and $X_1 = \mathbb{R}$. This contains I as a subcomplex.
2. $X = S^n$ can be made into a CW complex with X_k a point for $k < n$ and $X_n = S^n$. It can also be made into a CW complex in a different way, with $X_k = S^k$ for each k. With this CW structure, S^n has two k-cells for each $k \leq n$ (corresponding to the two hemispheres of $S^k \setminus S^{k-1}$), and S^k is a subcomplex of S^n.
3. S^∞/C_n, $U(n)$, $G_n(\mathbb{C}^m)$.

Proposition 15.4. A subset of a CW complex is compact iff it is contained in a finite subcomplex.

Proposition 15.5. The product of two CW complexes is a CW complex in a natural way.

Definition 15.6. A map $f\colon X \to Y$ of CW complexes is *cellular* if $f(X_n) \subseteq Y_n$ for all n. Two cellular maps $f_0, f_1\colon X \to Y$ are *cellularly homotopic* iff there is a cellular map $h\colon I \times X \to Y$ with $h(t,x) = f_t(x)$ for $t \in \{0,1\}$.

Theorem 15.7.

1. Any map $f_0\colon X \to Y$ is homotopic to a cellular map $f_1\colon X \to Y$, by a homotopy f_t say.
2. If W is a subcomplex of X and $f_0|_W$ is already cellular, then there exists a homotopy as above with $f_t|_W = f_0|_W$ for all t.
3. If two cellular maps $f_0, f_1\colon X \to Y$ are homotopic, then they are cellularly homotopic.

Theorem 15.8. If $f\colon X \to Y$ is an n-connected map and Z is a CW-complex with $\dim(Z) \leq n$, then the induced map $f_*\colon [Z,X] \to [Z,Y]$ is surjective. If $\dim(Z) < n$ then f_* is bijective. If f is a weak equivalence and Z is any CW complex then f_* is bijective.

Corollary 15.9. If X and Y are CW complexes and $f\colon X \to Y$ is a weak equivalence then it is a homotopy equivalence.

Theorem 15.10. For any space Y, there is a CW complex X and a weak equivalence $X \to Y$.

Definition 15.11. Let X be a CW complex, and write $W_nX = H_*(X_n, X_{n-1})$, which is the free Abelian group on the n-cells of X. The boundary map for the pair (X_n, X_{n-1}) together with the projection for the pair (X_{n-1}, X_{n-2}) gives a homomorphism

$$d_n\colon W_nX = H_n(X_n, X_{n-1}) \xrightarrow{\partial} H_{n-1}X_{n-1} \xrightarrow{\pi} H_{n-1}(X_{n-1}, X_{n-2}) = W_{n-1}X.$$

It is easily seen that $d_{n-1} \circ d_n = 0$, so we get a chain complex (W_*, d_*). This is called the *cellular chain complex* of X. It is functorial for cellular maps of CW complexes.

Theorem 15.12. For any CW complex X, there are natural isomorphisms

$$H_*(W_*(X) \otimes A) = H_*(X;A)$$
$$H^*(\mathrm{Hom}(W_*(X), A)) = H^*(X;A).$$

16. Stable homotopy theory

Let Z and W be finite CW complexes. By Corollary 14.3, the map $\Omega^n\Sigma^n W \to \Omega^{n+1}\Sigma^{n+1}W$ is $(n-1)$-connected. It follows that for large n, the natural map

$$\Sigma\colon [\Sigma^n Z, \Sigma^n W] = [Z, \Omega^n\Sigma^n W] \to [Z, \Omega^{n+1}\Sigma^{n+1}W] = [\Sigma^{n+1}Z, \Sigma^{n+1}W]$$

is an isomorphism. This justifies the following definition:

Definition 16.1. We write $\{Z, W\} = [\Sigma^n Z, \Sigma^n W]$ for $n \gg 0$. For some purposes it is more convenient to think of this as $\varinjlim_n [\Sigma^n Z, \Sigma^n W]$, but this comes to the same thing.

It is easy to see that the natural map $\Sigma\colon \{Z, W\} \to \{\Sigma Z, \Sigma W\}$ is an isomorphism. We can construct a category whose objects are finite CW complexes and whose morphism sets are the sets $\{Z, W\}$. The above observation shows that Σ acts bijectively on maps; it is convenient to arrange things such that Σ acts bijectively on objects as well. We thus make the following definition.

Definition 16.2. A finite spectrum is a pair (n, X), where $n \in \mathbb{Z}$ and X is a finite CW complex. This is to be thought of as $\Sigma^n X$, which nakes sense as a space if $n \geq 0$ but not if $n < 0$. We define

$$\{(n, X), (m, Y)\} = [\Sigma^{n+k}X, \Sigma^{m+k}Y] \qquad (k \gg 0),$$

which makes sense even when n and m are negative.

We can make the class $\mathcal{F}$ of finite spectra into a category, with morphism sets $\{(n, X), (m, Y)\}$. We define $\Sigma(n, X) = (n+1, X)$ and $\Sigma^{-1}(n, X) = (n-1, X)$. It is easy to make Σ and Σ^{-1} into functors and to show that they are mutually inverse equivalences of categories, and that there is a natural isomorphism $\Sigma(n, X) = (n, \Sigma X)$. It follows that $(n, X) = \Sigma^n(0, X)$. From now on we rarely use the notation (n, X) and write $\Sigma^n X$ instead.

There is a well-developed theory of infinite spectra (which is very useful even if one is primarily interested in finite complexes) but it takes quite a lot more work to set up.

Note that $[\Sigma^n Z, \Sigma^n W]$ is an Abelian group for $n > 1$, and thus that $\{Z, W\}$ has a natural structure as an Abelian group. One can check that the composition map $\{Z, W\} \times \{W, V\} \to \{Z, V\}$ is bilinear, so $\mathcal{F}$ is a preadditive category. One can also check that the wedge gives a coproduct on $\mathcal{F}$, so $\mathcal{F}$ is an additive category.

We can extend the definition of the smash product to $\mathcal{F}$ by defining $(n, X) \wedge (m, Y) = (n+m, X \wedge Y)$. (Some care with signs is necessary when defining smash product of morphisms.) One can show that this smash product is bilinear, so it can be thought of as a sort of tensor product of spectra.

This makes $\mathcal{F}$ look a bit like the category of finite-dimensional vector spaces over a field. We next study the analog of duality for vector spaces. Given a finite CW complex X, we can embed X as a proper subspace of S^N for large N. With care, things can be set up so that there is a finite CW complex Y which is a deformation retract of $S^N \setminus X$. (This will happen if X is embedded as a subcomplex in a simplicial triangulation of S^N, for example.) We then define $DX = \Sigma^{1-N}Y$, or more generally $D\Sigma^n X = \Sigma^{1-N-n}Y$. This construction D can be made into a contravariant functor with $D^2 = 1$. (If we had developed the theory of infinite spectra, we could define D in a more canonical and obviously functorial way; the work would be in proving that DX thus defined was homotopy equivalent to a finite spectrum.) There is a natural isomorphism

$$\{X, DY \wedge Z\} = \{X \wedge Y, Z\},$$

analogous to the isomorphism $\mathrm{Hom}(U, V^* \otimes W) = \mathrm{Hom}(U \otimes V, W)$ for vector spaces. We also have

$$\widetilde{H}_k DX = \widetilde{H}^{-k} X.$$

As an application of these ideas, we discuss Atiyah duality. Let M be a compact closed manifold, with tangent bundle τ. Write ϵ^n for the trivial bundle of dimension n over M. One can show that there is a surjective map $\epsilon^N \to \tau$ for some large N, with kernel ζ say. Write M^ζ for the one-point compactification of the total space of ζ (also called the *Thom space* of ζ) and $M^{-\tau}$ for the finite spectrum $\Sigma^{-N} M^\zeta$.

Theorem 16.3 (Thom)**.** If τ is orientable then $\widetilde{H}^* M^{-\tau}$ is a free module over $H^* M$ on one generator in degree $-n$.

Theorem 16.4 (Atiyah duality)**.** There is a natural isomorphism $D(M_+) = M^{-\tau}$ in $\mathcal{F}$.

Poincaré duality follows easily from these theorems.

For any cofibration $X \to Y \to Z$, there are doubly infinite long exact sequences

$$\ldots\{\Sigma^{-1}Z, W\} \leftarrow \{X, W\} \leftarrow \{Y, W\} \leftarrow \{Z, W\} \leftarrow \{\Sigma Z, W\}\ldots$$

and

$$\ldots\{W, \Sigma^{-1}Z\} \to \{W, X\} \to \{W, Y\} \to \{W, Z\} \to \{W, \Sigma Z\}\ldots.$$

The first of these follows easily from the unstable Puppe sequence for a fibration, but the second is a bit more surprising. It says that in the stable world, cofibrations behave like fibrations.

Appendix A. Compactly generated spaces

Let X be a topological space. We say that a subset $Y \subseteq X$ is k-closed if for each compact Hausdorff space K and each map $f\colon K \to X$, the preimage $f^{-1}Y$ is closed in X. It is easy to check that the k-closed sets are the closed sets for a new topology on X; we write kX for X equipped with this topology. One can check that when K is compact Hausdorff, the continuous maps $f\colon K \to X$ are the same as the continuous maps $f\colon K \to kX$, and thus that $k^2X = kX$. We say that X is *compactly generated (CG)* if $kX = X$.

If Y is a CG space then continuous maps $Y \to X$ are the same as continuous maps $Y \to kX$ (so the functor k is right adjoint to the inclusion of CG spaces in all spaces).

Given two spaces X and Y, we shall write $X \times_0 Y$ for the product space equipped with the usual product topology. This need not be CG even if X and Y are. We thus define $X \times Y = k(X \times_0 Y)$. This is the categorical product in the category of CG spaces. We will use this product without comment throughout these notes.

We say that a space X is *weakly Hausdorff (WH)* if for all compact Hausdorff spaces K and all maps $f\colon K \to X$ the image fK is closed in X. If X is a CG space, it turns out that X is WH iff the subset $\Delta_X = \{(x,x) \mid x \in X\}$ is closed in X^2. The product of two CGWH spaces is CGWH. All metric spaces, locally compact Hausdorff spaces, and CW complexes are CGWH. In particular, all topological manifolds, simplicial complexes, and real or complex algebraic varieties are CGWH.

If X is an arbitrary CG space, we let R be the smallest equivalence relation on X that is closed as a subspace of $X \times X$, and we define $hX = X/R$. One can check that hX is a CGWH space. Moreover, if Y is a CGWH space then any map $X \to Y$ factors uniquely through hX. Thus h is left adjoint to the inclusion of CGWH spaces in CG spaces.

Given two CG spaces X and Y, we write $C_0(X,Y)$ for the space of maps $X \to Y$, equipped with the compact-open topology. We also write $C(X,Y) = kC_0(X,Y)$. One can check that the functions

$$\eta\colon Y \to C(X, X \times Y) \qquad \eta(y)(x) = (x,y)$$

and

$$\epsilon\colon X \times C(X,Y) \to Y \qquad \epsilon(x,f) = f(x)$$

are continuous. It follows formally that the category of CGWH spaces is cartesian-closed and that all sorts of other functions are continuous.

Using the characterisation of CGWH spaces involving the diagonal, we see that $C(X,Y)$ is WH whenever Y is, and thus that the category of CGWH spaces is cartesian-closed.

Appendix B. Numerable bundles

Definition B.1. Let X be a space, and $\{U_i \mid i \in I\}$ an open covering of X. We say that $\{U_i\}$ is *locally finite* if every point $x \in X$ has an open neighbourhood V such that $\{i \in I \mid V \cap U_i \neq \emptyset\}$ is finite. We say that another covering $\{V_j \mid j \in J\}$ is a *refinement* of $\{U_i\}$ if for all $j \in J$ there exists $i \in I$ such that $V_j \subseteq U_i$. We say that X is *paracompact* if it is Hausdorff, and every open covering admits a locally finite refinement.

Theorem B.2. Locally compact Hausdorff spaces, metric spaces, and CW complexes are all paracompact.

Definition B.3. An open covering $\{U_i \mid i \in I\}$ of X is *numerable* if it is locally finite, and there are maps $\phi_i\colon X \to I$ such that $U_i = \{x \mid \phi_i(x) > 0\}$ for all i. A fibre bundle $q\colon E \to B$ is *numerable* if there is a numerable covering $\{B_i\}$ of B such that the restricted bundle $q^{-1}B_i \to B_i$ is trivial for each i.

Theorem B.4. If X is a paracompact space, then every open covering of X admits a numerable refinement. Thus, every fibre bundle over a paracompact space is numerable.

Theorem B.5. Let $q\colon E \to B$ be a map. Suppose that there exists a numerable covering $\{B_i\}$ of B such that the map $q^{-1}B_i \to B_i$ is a fibration for each i. Then q is a fibration.

Corollary B.6. Every numerable fibre bundle (and thus every fibre bundle over a paracompact base) is a fibration.