LOCAL FIBRATIONS

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In these notes we prove the following result.

Theorem 1. A numerable local fibration is a fibration. In particular, a numerable fibre bundle is a fibration. □

The relevant definitions are given below. After that, we will prove a number of lemmas, and we finish by deducing the theorem.

Definition 2. Let X be a space, and $\{U_i \mid i \in I\}$ an open covering of X. We say that $\{U_i\}$ is *locally finite* if every point $x \in X$ has an open neighbourhood V such that $\{i \in I \mid V \cap U_i \neq \emptyset\}$ is finite. We say that another covering $\{V_j \mid j \in J\}$ is a *refinement* of $\{U_i\}$ if for all $j \in J$ there exists $i \in I$ such that $V_j \subseteq U_i$. We say that X is *paracompact* if it is Hausdorff, and every open covering admits a locally finite refinement.

Definition 3. An open covering $\{U_i \mid i \in I\}$ of X is *numerable* if it is locally finite, and there are maps $\phi_i : X \to I$ such that $U_i = \{x \mid \phi_i(x) > 0\}$ for all i. A fibre bundle $q: E \to B$ is *numerable* if there is a numerable covering $\{B_i\}$ of B such that the restricted bundle $q^{-1}B_i \to B_i$ is trivial for each i.

Definition 4. We say that a map $q: E \to B$ is a *numerable local fibration* if there is a numerable covering $B = \bigcup_{i \in I} B_i$ such that the maps $q: q^{-1}B_i \to B_i$ are all fibrations.

Definition 5. We say that a map $q: E \to B$ is a *numerable fibre bundle* if there is a numerable covering $B = \bigcup_{i \in I} B_i$ and homeomorphisms $f_i : B_i \times F_i \to q^{-1}B_i$ (for some collection of spaces F_i) such that $qf_i(b, x) = b$. It is easy to see that a numerable fibre bundle is a numerable fibration.

For the rest of these notes, we assume that $q: E \to B$ is a numerable local fibration, with a numerable cover $\{B_i\}$ as above and given functions $f_i : B \to I$ with $B_i = f_i^{-1}(0, 1]$. We will prove a number of lemmas and then prove the theorem at the end of the notes.

Definition 6. Given a subset $W \subseteq \mathrm{Path}(B)$, we define

$$F(W) = \{(\omega, e) \in U \times E \mid \omega(1) = q(e)\} \subseteq \mathrm{Path}(q)$$
$$G(W) = \{(\omega, t, e) \in U \times I \times E \mid \omega(t) = q(e)\}.$$

A *lifting function* over W is a map $l: F(W) \to \mathrm{Path}(E)$ such that $l(\omega, e)(1) = e$ and $q \circ l(\omega, e) = \omega$. An *extended lifting function* for W is a map $m: G(W) \to \mathrm{Path}(E)$ such that $m(\omega, t, e)(t) = e$ and $q \circ m(\omega, t, e) = \omega$.

Thus, $q: E \to B$ is a fibration if and only if there is a lifting function over the whole of $\mathrm{Path}(B)$.

Lemma 7. Let B' be a subspace of B and $E' = q^{-1}B'$. Then the following are equivalent:

(a) $q: E' \to B'$ is a fibration.
(b) There is a lifting function over $\mathrm{Path}(B')$.
(c) There is an extended lifting function over $\mathrm{Path}(B')$.

Proof. It is immediate from the definitions that (a)⇔(b). If m is an extended lifting function over $\mathrm{Path}(B')$ then $l(\omega, e) = m(\omega, 1, e)$ is a lifting function, so (c)⇒(b). Conversely, suppose that l is a lifting function. Suppose that $(\omega, t, e) \in G(\mathrm{Path}(B'))$, so $\omega: I \to B'$ and $q(e) = \omega(t)$. We then define

$$\omega_+(s) = \omega(\max(s - 1 + t, 0))$$
$$\omega_-(s) = \omega(\min(1 + t - s, 1)).$$

Thus ω_+ sits at $\omega(0)$ for a while then runs forwards to reach $\omega(t)$ when $s = 1$, and ω_- sits at $\omega(1)$ for a while and then runs backwards to reach $\omega(t)$ when $s = 1$. We next define

$$m(\omega, t, e)(s) = \begin{cases} 0 \leq s \leq t & l(\omega_+, e)(s + 1 - t) \\ t \leq s \leq 1 & l(\omega_-, e)(1 + t - s). \end{cases}$$

One can check that this gives an extended lifting function. □

It follows that we can choose extended lifting functions m_i over $\mathrm{Path}(B_i)$.

Definition 8. Let $\underline{i} = (i_1, \ldots, i_r)$ be a sequence of indices. We write $\mathrm{len}(\underline{i}) = r$ and

$$W_{\underline{i}} = \{\omega \colon I \to B \mid \omega([(j-1)/r, j/r]) \subseteq B_{i_r} \text{ for } 1 \leq j \leq r\}.$$

We also define $f_{\underline{i}} \colon \mathrm{Path}(B) \to I$ by

$$f_{\underline{i}}(\omega) = \min_{1 \leq j \leq r} \min_{(j-1)/r \leq t \leq j/r} f_{i_j}(\omega(t)).$$

One can check that this is a continuous map $\mathrm{Path}(B) \to I$ and that $f_{i_j}(\omega) > 0$ if and only if $\omega \in W_{\underline{i}}$.

Lemma 9. The sets $W_{\underline{i}}$ form an open cover of $\mathrm{Path}(B)$.

Proof. It is easy to see that $W_{\underline{i}}$ is open. Suppose that $\omega \in \mathrm{Path}(B)$. For each $t \in I$ there is an index i_t and a number $\epsilon_t > 0$ such that $(t - \epsilon_t, t + \epsilon_t) \cap [0, 1] \subseteq \omega^{-1} B_{i_t}$. The open sets $U_t = (t - \epsilon_t/2, t + \epsilon_t/2)$ cover $[0, 1]$, so $[0, 1] = U_{t_1} \cup \ldots \cup U_{t_n}$ for some finite sequence $t_1, \ldots, t_n$. Write $\epsilon = \min(\epsilon_{t_1}, \ldots, \epsilon_{t_n})$. It is not hard to check that for any open interval (a, b) of length at most ϵ, we have $\omega((a, b)) \subseteq B_i$ for some i. Thus if we choose r such that $1/r < \epsilon$ then there is a sequence $\underline{i} = (i_1, \ldots, i_r)$ such that $\omega([(j-1)/r, j/r]) \subseteq B_{i_j}$ for all j and thus $\omega \in W_{\underline{i}}$. □

Lemma 10. There is an extended lifting function $m_{\underline{i}}$ over $W_{\underline{i}}$.

Proof. Consider a point $(\omega, t, e) \in G(W_{\underline{i}})$, so that $\omega \in W_{\underline{i}}$ and $q(e) = \omega(t)$. Write $r = \mathrm{len}(\underline{i})$ and let n be an integer such that $(n-1)/r \leq t \leq n/r$. There is only one such n unless t has the form m/r in which case n could be m or $m + 1$; one can check that the constructions below do not depend on which we take in that case. For $1 \leq j \leq r$ we define a path $\omega_j \colon I \to B_{i_j}$ by

$$\omega_j(s) = \omega(\max(\min(s, (j-1)/r), j/r),$$

so that ω_j is constant at $\omega((j-1)/r)$ until $s = (j-1)/r$, then it moves to $\omega(j/r)$ when $s = j/r$, then it sits there until $s = 1$. We define points $e_j \in E$ for $j = 0, \ldots, r$ inductively by

$$\begin{aligned} e_n &= m_{i_n}(\omega_n, t, e)(n/r) \\ e_{n-1} &= m_{i_n}(\omega_n, t, e)((n-1)/r) \\ e_{j+1} &= m_{i_{j+1}}(\omega_{j+1}, j/r, e_j)((j+1)/r) \qquad \text{for } n \leq j < r \\ e_{j-1} &= m_{i_{j-1}}(\omega_{j-1}, j/r, e_j)((j-1)/r) \qquad \text{for } 0 < j < n. \end{aligned}$$

We then define $\alpha \colon I \to E$ by

$$\alpha(s) = \begin{cases} m_{i_j}(\omega_j, j/r, e_j)(s) & \text{if } s \in [(j-1)/r, r] \text{ and } j < n \\ m_{i_n}(\omega_n, t, e)(s) & \text{if } s \in [(n-1)/r, r] \\ m_{i_j}(\omega_j, (j-1)/r, e_{j-1})(s) & \text{if } s \in [(j-1)/r, r] \text{ and } n < j. \end{cases}$$

One can check that this is well-defined and continuous, and that the assignment $m_{\underline{i}}(\omega, t, e) = \alpha$ gives an extended lifting function for $W_{\underline{i}}$. □

We have now covered $\mathrm{Path}(B)$ by open sets $W_{\underline{i}}$ such that there is an extended lifting function defined over each $\underline{i}$. We next need a way to glue together (extended) lifting functions over open sets V and W to get one over $V \cup W$. We first make a preliminary definition which does something a bit weaker.

Definition 11. Let W be a subset of $\mathrm{Path}(B)$, let l be a lifting function over W and let m be an extended lifting function over W. For each $s \in [0,1]$ we define a "merged" lifting function by

$$M_s(l,m)(\omega,e)(t) = \begin{cases} l(\omega,e)(t) & \text{if } s \leq t \leq 1 \\ m(\omega,s,l(\omega,e)(s))(t) & \text{if } 0 \leq t \leq s. \end{cases}$$

In other words, we lift the section of ω from $t = s$ up to $t = 1$ using l. This gives us a lift $l(\omega,e)(s)$ of $\omega(s)$ and we feed this into m to lift the section of ω from $t = 0$ up to $t = s$. It is easy to check that $M_s(l,m)$ is indeed a lifting function. Moreover, $M_0(l,m) = l$ and $M_1(l,m) = m(-,1,-)$.

Lemma 12. Let V, W be open subsets of $\mathrm{Path}(B)$. Suppose that we have functions $g, f\colon \mathrm{Path}(B) \to I$ such that $g^{-1}(0,1] = V$ and $f^{-1}(0,1] = W$. Suppose that we also have a lifting function l over V and an extended lifting function m over W. Then there is a lifting function over $V \cup W$ which agrees with l over $V \setminus W$.

Proof. Define $h\colon V \cup W \to I$ by $h = f/(f+g)$, and note that $\omega \in V$ iff $h(\omega) < 1$ and $\omega \in W$ iff $h(\omega) > 0$. Thus if $h(\omega) \leq 1/3$ we have $\omega \in V$, if $1/3 \leq h(\omega) \leq 2/3$ we have $h(\omega) \in U \cap V$, and if $h(\omega) \geq 2/3$ we have $\omega \in W$. We can thus define a lifting function over $V \cup W$ by

$$n(\omega,e) = \begin{cases} l(\omega,e) & \text{if } 0 \leq h(\omega) \leq 1/3 \\ M_{3h(\omega)-1}(l,m)(\omega,e) & \text{if } 1/3 \leq h(\omega) \leq 2/3 \\ m(\omega,1,e) & \text{if } 2/3 \leq h(\omega) \leq 1. \end{cases}$$

If $\omega \in V \setminus W$ then $h(\omega) = 0$ and so $n(\omega,e) = l(\omega,e)$ as claimed. □

This gives us a lifting function defined over any finite union of sets of the form $W_{\underline{i}}$. Unfortunately, we need to work a bit harder to get a lifting function defined over an infinite union of such sets.

Lemma 13. For any r, the collection of sets $\{W_{\underline{i}} \mid \mathrm{len}(\underline{i}) = r\}$ is locally finite.

Proof. Consider a path $\omega \in \mathrm{Path}(B)$. We need to produce a neighbourhood V of ω in $\mathrm{Path}(B)$ such that the set $S = \{(i_1,\dots,i_r) \mid V \cap W_{\underline{i}} \neq \emptyset\}$ is finite. We know that $\{B_i\}$ is locally finite, so for each $j = 1,\dots,r$ there is a neighbourhood V_j of $\omega(j/r)$ such that the set $S_j = \{i \mid V_j \cap B_i \neq \emptyset\}$ is finite. Write $V = \{\alpha \in \mathrm{Path}(B) \mid \alpha(j/r) \in V_j \text{ for } j = 1,\dots,r\}$. This is a neighbourhood of ω, and it is easy to see that for this V we have $S \subseteq S_1 \times \ldots \times S_r$, so S is finite as required. □

Corollary 14. The function $g_r(\omega) = r\sum_{\mathrm{len}(\underline{i})<r} f_{\underline{i}}(\omega)$ is finite and continuous.

Definition 15. We now define

$$f'_{\underline{i}} = \min(\max(0, f_{\underline{i}} - g_{\mathrm{len}(\underline{i})}), 1)\colon \mathrm{Path}(B) \to I$$

and

$$W'_{\underline{i}} = \{\omega \mid f'_{\underline{i}}(\omega) > 0\}.$$

It is easy to check that $W'_{\underline{i}} \subseteq W_{\underline{i}}$, so there is an extended lifting function over $W'_{\underline{i}}$.

Lemma 16. The collection of sets $\{W'_{\underline{i}}\}$ is a numerable covering of $\mathrm{Path}(B)$.

Proof. Consider a path $\omega \in \mathrm{Path}(B)$. Choose a set $W_{\underline{i}}$ such that $\omega \in W_{\underline{i}}$ with $r = \mathrm{len}(\underline{i})$ as small as possible. We then have $g_r(\omega) = 0$ and thus $f'_{\underline{i}}(\omega) = f_{\underline{i}}(\omega) > 0$, so $\omega \in W'_{\underline{i}}$. This means that the sets $W'_{\underline{i}}$ form an open cover of $\mathrm{Path}(B)$. We next need to show that our collection is locally finite. To do this, choose $N > r$ such that $1/N < f_{\underline{i}}(\omega)$, and note that $g_N(\omega) > 1$. Set $V = \{\alpha \in \mathrm{Path}(B) \mid g_N(\omega) > 1\}$, which is an open neighbourhood of ω. If $\alpha \in V$ then for $m \geq N$ we have $g_m(\alpha) > 1$ and thus $f'_{\underline{j}}(\alpha) = 0$ whenever $\mathrm{len}(\underline{j}) = m$. Thus, if V meets $W'_{\underline{j}}$ then we must have $\mathrm{len}(\underline{j}) < N$. We know that the collection $\{W_{\underline{j}} \mid \mathrm{len}(\underline{j}) < N\}$ is locally finite, so there is a neighbourhood V' of ω which meets only finitely many of the $W_{\underline{j}}$'s with $\mathrm{len}(\underline{j}) < N$. It follows that $V \cap V'$ meets only finitely many of the $W'_{\underline{j}}$'s. Thus $\{W'_{\underline{i}}\}$ is locally finite. We also have a map $f'_{\underline{i}}\colon \mathrm{Path}(B) \to I$ with $(f'_{\underline{i}})^{-1}(0,1] = W'_{\underline{i}}$. This shows that $\{W'_{\underline{i}}\}$ is a numerable covering. □

Proof of Theorem 1. We will write the proof in terms of transfinite recursion; it can be rewritten to use Zorn's lemma instead if you prefer. We may assume that the collection of tuples $\underline{i}$ is well-ordered. (If the collection of indices i is already well-ordered, we can order the tuples lexicographically; if not, we can appeal to the axiom of choice to get a random well-ordering of the tuples.) Thus, after a slight change of notation we have a locally finite covering $\{W'_\alpha\}$ indexed by the ordinals $\alpha < \kappa$ for some fixed ordinal κ. We also have functions $f'_\alpha\colon \text{Path}(B) \to I$ with $W'_\alpha = (f'_\alpha)^{-1}(0,1]$ and extended lifting functions m_α over W'_α. Because the family is locally finite, the function $g_\alpha = \sum_{\beta<\alpha} f'_\alpha$ is continuous. We write $V_\alpha = \bigcup_{\beta<\alpha} W'_\beta = (g_\alpha)^{-1}(0,1]$. We next define lifting functions l_α over V_α by transfinite recursion, such that when $\alpha < \beta$ and $\omega \in V_\alpha$ we have $l_\alpha(\omega, e) = l_\beta(\omega, e)$ unless $\omega \in W'_\gamma$ for some $\gamma \in [\alpha, \beta)$. (Note that there are only finitely many ordinals γ such that $\omega \in W'_\gamma$, so the lift $l_\alpha(\omega, e)$ will only change a finite number of times as α varies.)

As $V_0 = \emptyset$, the recursion starts. Given a successor ordinal $\alpha+$, we feed $l_\alpha, m_\alpha, f_\alpha$ and g_α into Lemma 12 to define a lifting function $l_{\alpha+}$ on $V_\alpha \cup W'_\alpha = V_{\alpha+}$ which agrees with l_α on $V_\alpha \setminus W'_\alpha$. Now consider a limit ordinal λ, and a point $(\omega, e) \in F(V_\lambda)$. Because $\{W'_\alpha\}$ is locally finite, we can choose a neighbourhood U of ω such that $S = \{\alpha \mid U \cap W'_\alpha \neq \emptyset\}$ is finite. Choose any ordinal α with $\max(S) < \alpha < \lambda$ and define $l_\lambda(\omega, e) = l_\alpha(\omega, e)$. This is independent of the choice of α because $l_\alpha(\omega, e) = l_\beta(\omega, e)$ unless $\omega \in W'_\gamma$ for some $\gamma \in [\alpha, \beta)$. Note that l_λ actually agrees with l_α on the neighbourhood $V_\alpha \cap U$ of ω and l_α is continuous so l_λ is continuous at (ω, e). As (ω, e) was arbitrary, we see that l_λ is continuous.

At the end of the recursion we have a lifting function defined over $V_\kappa = \text{Path}(B)$, as required. □