FIBRATIONS AND COFIBRATIONS

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Proposition 1. Let $j\colon W \to X$ be a map. The following are equivalent:

(a) j is a cofibration.
(b) j is left orthogonal to all maps of the form $p_1\colon \operatorname{Path}(E) \to E$.
(c) j has the homotopy extension property: for any map $f\colon X \to E$ and any homotopy $g_t\colon W \to E$ ending with $g_1 = fj$, there is a homotopy $h_t\colon X \to E$ extending g_t (in the sense that $h_t \circ j = g_t$) and ending with $h_1 = f$.

Proof. (a)⇒(c): Let j be a cofibration, so there is a retraction $r\colon I \times X \to I \times W \cup_W X$. Given maps $f\colon X \to E$ and $g\colon I \times W \to E$ as in (c) we define a map $k\colon I \times W \cup_W X \to E$ by $k(t, w) = g(t, w)$ on $I \times W$ and $k(x) = f(x)$ on X; this is consistent with the equivalence relation $(1, w) = jw$ because $g(1, w) = fj(w)$. We then define $h = kr\colon I \times X \to X$, and check that this is as required in (c).

(c)⇒(a): Suppose that (c) holds. Take $E = I \times W \cup_W X$, and let $f\colon X \to E$ and $g\colon I \times W \to E$ be the obvious maps. We then get a map $h\colon I \times X \to E$ such that $h(1, x) = f(x)$ and $h(t, jw) = (t, w)$ when $w \in W$. It follows that h is a retraction onto $I \times W \cup_W X$, so j is a cofibration.

(b)⇔(c): A square of the form

$$\begin{array}{ccc} W & \xrightarrow{g^\#} & \operatorname{Path}(E) \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle p_1} \\ X & \xrightarrow[f]{} & E \end{array}$$

is the same thing as a pair of maps $f\colon X \to E$, $g\colon I \times W \to E$ such that $g(1, w) = fj(w)$, via the usual translation $g(t, w) = g^\#(w)(t)$. A fill in map $h^\#\colon X \to \operatorname{Path}(E)$ with $p_1 h^\# = f$ and $h^\# j = g^\#$ is the same as a map $h\colon I \to E$ such that $h(1, x) = f(x)$ and $h(t, jw) = g(t, w)$. Thus (b) is just a translation of (c). □

Proposition 2. Let $q\colon E \to B$ be a map. The following are equivalent:

(a) q is a fibration.
(b) q is right orthogonal to all maps of the form $i_1\colon X \to I \times X$.
(c) q has the homotopy lifting property: given a homotopy $g_t\colon X \to B$ and a map $f\colon X \to E$ which lifts g_1 (in the sense that $qf = g_1$) there is a lifted homotopy $h_t\colon X \to E$ with $qh_t = g_t$ and $h_1 = f$.

Proof. Exercise. □

Definition 3. A closed subspace $W \subseteq X$ is a *neighbourhood deformation retract (NDR)* if there exist maps $u\colon X \to I$ and $h\colon I \times X \to X$ such that

(a) $W = u^{-1}\{0\}$.

(b) $h_1 = 1_X$.
(c) $h_t|_W = 1_W$ for all $t \in I$.
(d) $h_0(x) \in W$ for all $x \in X$ such that $u(x) < 1$.

We say that (u, h) is a *representation* of W as an NDR. We say that W is a *deformation retract (DR)* if we can choose h such that $h_1(X) \subseteq W$. This holds automatically if $u(x) < 1$ for all x, and conversely if $h_1(X) \subseteq W$ then we can replace u by $u/2$ and assume that $u < 1$ everywhere. Note also that in this case h_1 is a retraction of X onto W.

Proposition 4. If $W \subseteq X$ and $Y \subseteq Z$ are NDR's, then so is $W \times Z \cup X \times Y \subseteq X \times Z$. Moreover, this is a DR if either $W \subseteq X$ or $Y \subseteq Z$ is.

Proof. Let (u, h) and (v, k) represent W and Y as NDR's. Write $T = W \times Z \cup X \times Y$. Define $w\colon X \times Z \to I$ by $w(x, z) = u(x)v(z)$; it is clear that $w^{-1}\{0\} = T$. Define $q\colon I \times X \times Z \to X \times Z$ by

$$q(t, x, z) = \begin{cases} (x, z) & \text{if } ux = vz = 0 \\ (h(t, x), k(1 - (1 - t)ux/vz, z)) & \text{if } vz \geq ux \text{ and } vz > 0 \\ (h(1 - (1 - t)vz/ux, x), k(t, z)) & \text{if } ux \geq vz \text{ and } ux > 0. \end{cases}$$

We need to show that this is well-defined continuous. It is well-defined because the second and third clauses both give $(h(t, x), k(t, z))$ when $ux = vz > 0$. The set where $ux > 0$ is the union of the two relatively closed sets where the second and third clauses apply. It follows easily that q is continuous on $\{(t, x, z) \mid ux > 0\}$, and similarly on $\{(t, x, z) \mid vz > 0\}$. All that is left is to check that q is continuous at points (t, x, z) where $ux = vz = 0$. Note that this implies that $x \in W$ and $z \in Y$ and $q(t, x, z) = (x, z)$. Let U and V be neighbourhoods of x and z in X and Z. Write $U' = \{x' \in X \mid h(t, x) \in U \text{ for all } t\}$. As $x \in W$ we have $h(t, x) = x$ for all t and thus $x \in U'$. We can also describe U' as the preimage of $C(I, U)$ under the map $h^{\#}\colon X \to C(I, X)$ that is adjoint to $h\colon I \times X \to X$. Note that $C(I, U)$ is open in $C(I, X)$ (even in the compact-open topology, and *a fortiori* in the standard topology), so U' is open in X. It is clear that $q(I \times U' \times V') \subseteq U \times V$. It follows that both components of q are continuous, and thus that q is continuous as required.

It is easy to check that $q_1 = 1_{X \times Z}$. Suppose that $(x, z) \in W \times Z$, so $ux = 0$. If $vz = 0$ then $q(t, x, z) = (x, z)$ by the first clause in the definition of q. If $vz > 0$ then the second clause applies and $q(t, x, z) = (h(t, x), k(1, z))$ but $k_1 = 1_Z$ and $h_t(x) = x$ because $x \in W$, so $q(t, x, z) = (x, z)$. This shows that $q(t, x, z) = (x, z)$ when $(x, z) \in W \times Z$, and the same holds when $(x, z) \in X \times Y$ by a similar argument. Thus $q_t|_T = 1_T$ for all t.

Finally, suppose that $w(x, z) < 1$. We either have $ux \leq vz$ and $ux < 1$ or $vz \leq ux$ and $vz < 1$, without loss of generality the former. As $ux < 1$ we have $h(0, x) \in W$, and as $ux \leq vz$ either the first or second clause in the definition of $q(t, x, z)$ applies. Either way we see easily that $q(0, x, z) \in T$. This proves that (w, q) represents $T \subseteq X \times Z$ as an NDR.

If W is a DR of X then we may assume that $u < 1$ everywhere. It follows immediately that $w < 1$ everywhere and thus that T is a DR of $X \times Z$. Clearly this also applies if Y is a DR of Z. □

Proposition 5. A map $j\colon W \to X$ is a cofibration if and only if it is a closed inclusion and jW is an NDR of X.

Proof. First suppose that j is a closed inclusion (so we can harmlessly think of W as a subspace of X) and that W is an NDR of X. It is easy to see that $\{1\}$ is a DR of I (take $u(s) = 1 - s$ and $h(t, s) = 1 - t + ts$). It follows from Proposition 4 that $1 \times X \cup I \times W$ is a DR of $I \times X$, so there is a map $r = h_1 : I \times X \to 1 \times X \cup I \times W$ which is the identity on $1 \times X \cup I \times W$. As j is a closed inclusion, one can check that $\mathrm{Cyl}(j)$ is just the space $1 \times X \cup I \times W$, so this map r is precisely what we need to show that j is a cofibration.

Conversely, suppose that $j : W \to X$ is a cofibration. One can check from the definitions that the map $W \xrightarrow{i_1} I \times W \to \mathrm{Cyl}(j)$ is always a closed inclusion. As j is a cofibration, the evident map $\mathrm{Cyl}(j) \to I \times X$ has a left inverse. As everything is weakly Hausdorff, it follows that $\mathrm{Cyl}(j) \to I \times X$ is also a closed inclusion, and thus that the composite map $W \to I \times X$ (sending w to $(1, j(w))$) is also a closed inclusion. It is not hard to conclude that j is a closed inclusion. We may therefore harmlessly think of W as a subspace of X, and of $\mathrm{Cyl}(j)$ as $1 \times X \cup I \times W$. The retraction $r : I \times X \to 1 \times X \cup I \times W \subseteq I \times X$ thus has the form $r(t, x) = (v(t, x), h(t, x))$, where $v : I \times X \to I$ and $h : I \times X \to X$. We define $u(x) = v(0, x)$; one can check that (u, h) represents W as an NDR of X. □

Corollary 6. A smashout of cofibrations is a cofibration.

Proof. Let $j : W \to X$ and $k : Y \to Z$ be cofibrations. Then we can think of j and k as inclusions of subspaces, and their smashout is just the inclusion $W \times Z \cup X \times Y \to X \times Z$, so the claim follows from Proposition 4. □

Proposition 7. A map $j : W \to X$ is an acyclic cofibration if and only if it is a closed inclusion and jW is a DR of X.

Proof. By proposition 5, we may assume that j is the inclusion of a closed subspace and that W is an NDR of X, represented by (u, h) say. If W is a DR we may assume that $h_1(X) = W$, and it is easy to check that $h_1 : X \to W$ is a homotopy inverse for j, so that j is an acyclic cofibration.

For the converse, suppose that j is an acyclic cofibration. We then have a homotopy inverse $f : X \to W$ with $fj \simeq 1_W$ and $jf \simeq 1_X$. After extending the homotopy $fj \simeq 1_W$ over X (using the homotopy extension property of cofibrations) we may assume that $fj = 1_W$. Let $g_t : X \to X$ be a homotopy with $g_0 = 1_X$ and $g_1 = jf$. Define $P = \{0, 1\} \times X \cup I \times W$ and $Q = I \times X$. It is easy to see that $\{0, 1\} \subset I$ is an NDR, so Proposition 4 tells us that P is an NDR of Q, and thus that $1 \times Q \cup I \times P$ is a retract of $I \times Q = I^2 \times X$. We define a map $h : 1 \times Q \cup I \times P \to X$ by

$$
\begin{aligned}
h(s, 0, x) &= g(s, jf(x)) \\
h(1, t, x) &= g(1 - t, x) \\
h(s, 1, x) &= x \\
h(s, t, w) &= g(s(1 - t), j(w)) \qquad \text{for } w \in W.
\end{aligned}
$$

Note that the first and second clauses are consistent because $g_0 = jf$ and $fj = 1_W$ so $g_0 jf = jfjf = jf$. All other consistency checks are left to the reader. Because $1 \times Q \cup I \times P$ is a retract of $I^2 \times X$, we can extend h over all of $I^2 \times X$ (just compose with the retraction). Having done this, we define $k(t, x) = h(0, t, x)$, so that $k : I \times X \to X$. We find that $k(1, x) = x$ for all x, that $k(t, w) = w$ for all t and all $w \in W$, and that $k(0,x) = f(x) \in W$ for all x. It follows that (u,k) represents W as a DR of X. □

Corollary 8. The smashout of a cofibration and an acyclic cofibration is an acyclic cofibration.

Proof. This is immediate from Propositions 4 and 7. □

Proposition 9. If $j\colon W \to X$ is an acyclic cofibration then there is a diagram

$$\begin{array}{ccccc} W & \xrightarrow{j} & X & \xrightarrow{r} & W \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle j} \\ X & \xrightarrow[k]{} & I \times X & \xrightarrow[s]{} & X, \end{array}$$

in which $rj = 1_W$ and $sk = 1_X$. In other words, the map j is a retract of the map i_1.

Proof. We may assume that W is a closed subspace of X. Choose (u,h) representing W as a DR of X. Define $g\colon I \times X \to X$ by $g(t,x) = h(\max(t/ux, 1), x)$. This is clearly continuous on $I \times (X \setminus W)$. Suppose that $(t,w) \in I \times W$ (so that $g(t,w) = w$) and that U is an open neighbourhood of w in X. Write $U' = \{x \in X \mid h(I \times \{x\}) \subseteq U\}$. As in the proof of Proposition 4, we see that this is an open neighbourhood of w. Clearly $g(I \times U') \subseteq U$, and thus g is continuous at (t,w). This shows that g is continuous everywhere. One can check that (u,g) represents W as a DR of X, and that $g(t,x) = x$ whenever $t \geq ux$.

Now define

$$\begin{aligned} k(x) &= (1 - u(x), x) \\ r(x) &= g(0,x) \\ s(t,x) &= g(1-t, x). \end{aligned}$$

One can check that the diagram commutes. □

Corollary 10. If $j\colon W \to X$ is an acyclic cofibration and $q\colon E \to B$ is a fibration then j is left orthogonal to q.

Proof. Suppose we are given a diagram of the following form:

$$\begin{array}{ccc} W & \xrightarrow{f} & E \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow[g]{} & B. \end{array}$$

Choose maps k, r, s as in Proposition 9. The homotopy lifting property tells us that there is a map $h\colon I \times X \to E$ making the following diagram commute:

$$\begin{array}{ccc} X & \xrightarrow{fr} & E \\ {\scriptstyle i_1}\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle q} \\ I \times X & \xrightarrow[gs]{} & B. \end{array}$$

It follows that we can add the map $hk\colon X \to E$ to the original diagram and it will still commute. This means that j is orthogonal to q. □

Proposition 11. If $q\colon E \to B$ is an acyclic fibration then it is homotopy equivalent over B to B.

Proof. As q is a homotopy equivalence, there is a map $e\colon B \to E$ such that $qe \simeq 1_B$ and $eq \simeq 1_E$. After lifting the homotopy $qe \simeq 1_B$ (using the homotopy lifting property of fibrations) we may assume that $qe = 1_B$. Choose a homotopy $g_t\colon E \to E$ with $g_0 = 1_E$ and $g_1 = eq$. Write $J = \{1\} \times I \cup I \times \{0, 1\} \subset I^2$, and define maps $n\colon J \times E \to E$ and $m\colon I^2 \times E \to B$ by

$$\begin{aligned} n(s, 0, x) &= eqg(s, x) \\ n(1, t, x) &= g(1 - t, x) \\ n(s, 1, x) &= x \\ m(s, t, x) &= qg(s(1 - t), x). \end{aligned}$$

One can check that the following diagram commutes:

$$\begin{array}{ccc} J \times E & \xrightarrow{n} & E \\ \downarrow & & \downarrow{\scriptstyle q} \\ I^2 \times E & \xrightarrow[m]{} & B. \end{array}$$

The inclusion $J \times E \rightarrowtail I^2 \times E$ is an acyclic cofibration, so it is orthogonal to q by Proposition 10. Thus, there is a map $l\colon I^2 \times E \to E$ filling in the square. Define $h\colon I \times E \to E$ by $h(t, x) = l(0, t, x)$. Then $h(0, x) = eq(x)$ and $h(1, x) = x$ and $qh(t, x) = q(x)$, so h is a homotopy over B between eq and 1_E, as required. □

Corollary 12. If $q\colon E \to B$ is an acyclic fibration, then for each $b \in B$ the fibre $q^{-1}\{b\}$ is contractible.

Proposition 13. If $j\colon W \to X$ is a cofibration and $q\colon E \to B$ is an acyclic fibration, then j is left orthogonal to q.

Proof. Let e and h be as in the proof of Proposition 11. Given a diagram of the form

$$\begin{array}{ccc} W & \xrightarrow{f} & E \\ {\scriptstyle j}\big\downarrow & & \big\downarrow{\scriptstyle q} \\ X & \xrightarrow[g]{} & B, \end{array}$$

we consider the diagram

$$\begin{array}{ccc} I\times W\cup 0\times X & \xrightarrow{k} & E \\ \big\downarrow & & \big\downarrow{\scriptstyle q} \\ I\times X & \xrightarrow[g\circ\mathrm{proj}]{} & B. \end{array}$$

The left hand vertical map is an acyclic cofibration, which is orthogonal to q by Proposition 10. There is thus a map $l\colon I\times X\to E$ filling in the square. One can check that the map $x\mapsto l(1,x)$ fills in the original square. □

Proposition 14. We have $\mathrm{acf}^{\perp}=\mathrm{fib}$ and $\mathrm{cof}^{\perp}=\mathrm{afb}$.

Proof. We have seen in Propositions 10 and 13 that $\mathrm{acf}\perp\mathrm{fib}$ and $\mathrm{cof}\perp\mathrm{afb}$, so that $\mathrm{fib}\subseteq\mathrm{acf}^{\perp}$ and $\mathrm{afb}\subseteq\mathrm{cof}^{\perp}$. Suppose that $q\in\mathrm{acf}^{\perp}$. Recall that $\mathrm{Path}(q)=\{(\omega,e)\in\mathrm{Path}(B)\times E\mid\omega(1)=q(e)\}$. We define maps $f\colon\mathrm{Path}(q)\to E$ and $g\colon I\times\mathrm{Path}(q)\to B$ by $f(\omega,e)=e$ and $g(t,\omega,e)=\omega(t)$. This gives a diagram as follows:

$$\begin{array}{ccc} \mathrm{Path}(q) & \xrightarrow{f} & E \\ {\scriptstyle i_1}\big\downarrow & & \big\downarrow{\scriptstyle q} \\ I\times\mathrm{Path}(q) & \xrightarrow[g]{} & B. \end{array}$$

As i_1 is clearly an acyclic cofibration, there is a map $m\colon I\times\mathrm{Path}(q)\to E$ filling in the square. One can check that the adjoint map $l=m^{\#}\colon\mathrm{Path}(q)\to\mathrm{Path}(E)$ (defined by $l(\omega,e)(t)=m(t,\omega,e)$) is a path-lifting function for q, so q is a fibration.

Now suppose that $q\in\mathrm{cof}^{\perp}$. The above shows that q is a fibration; we need to show that it is also a homotopy equivalence. By filling in the square on the left below, we get a map $e\colon B\to E$ with $qe=1_B$. We then fill in the right hand square (in which $g(0,x)=eq(x)$ and $g(1,x)=x$) to get a homotopy $h\colon I\times E\to E$ over B between eq and 1_E, as required.

$$\begin{array}{ccc} \emptyset & \longrightarrow & E \\ \big\downarrow & & \big\downarrow{\scriptstyle q} \\ B & \xrightarrow[1]{} & B \end{array} \qquad \begin{array}{ccc} \{0,1\}\times E & \xrightarrow{g} & E \\ \big\downarrow & & \big\downarrow{\scriptstyle q} \\ I\times E & \xrightarrow[q\circ\mathrm{proj}]{} & B. \end{array}$$

□

Proposition 15. If $j\colon W\to X$ is a cofibration and $q\colon E\to B$ is a fibration then the crossmap $F(j,q)\colon C(X,E)\to C(W,E)\times_{C(W,B)}C(X,B)$ is a fibration. If j or q is acyclic then so is $F(j,q)$.

Proof. Let i be an acyclic cofibration. Then $i\Box j$ is an acyclic cofibration and thus orthogonal to q, so $F(i\Box j, q)$ is surjective. However, $F(i, F(j,q)) = F(i\Box j, q)$ so $F(i, F(j,q))$ is surjective, so i is orthogonal to $F(j,q)$. This holds for all $i \in \mathrm{acf}$, so $F(j,q) \in \mathrm{acf}^{\perp} = \mathrm{fib}$ as claimed. A similar argument shows that if $j \in \mathrm{acf}$ or $q \in \mathrm{afb}$ then $F(j,q) \in \mathrm{cof}^{\perp} = \mathrm{fib}$. □

Corollary 16. If $j\colon W \to X$ is a cofibration and E is any space, then the restriction map $j^*\colon C(X,E) \to C(W,E)$ is a fibration. If j is acyclic then so is j^*.

Proof. Apply Proposition 15 to the map $q\colon E \to 0$. □

Proposition 17. We have $\mathrm{acf} = {}^{\perp}\,\mathrm{fib}$ and $\mathrm{cof} = {}^{\perp}\,\mathrm{afb}$.

Proof. We already know that $\mathrm{acf} \subseteq {}^{\perp}\,\mathrm{fib}$ and $\mathrm{cof} \subseteq {}^{\perp}\,\mathrm{afb}$. Suppose that $j \in {}^{\perp}\,\mathrm{afb}$. Let $f\colon I \times W \to \mathrm{Cyl}(j)$ and $g\colon X \to \mathrm{Cyl}(j)$ be the evident maps, so that $f(1,w) = gj(w)$. As usual, we write $f^{\#}\colon W \to \mathrm{Path}\,\mathrm{Cyl}(j)$ for the adjoint map, defined by $f^{\#}(w)(t) = f(t,w)$. This gives a commutative square as follows:

$$\begin{array}{ccc} W & \xrightarrow{f^{\#}} & \mathrm{Path}\,\mathrm{Cyl}(j) \\ {\scriptstyle j}\big\downarrow & & \big\downarrow{\scriptstyle p_1} \\ X & \xrightarrow[g]{} & \mathrm{Cyl}(j). \end{array}$$

We know from Corollary 16 that the map p_1 is an acyclic fibration and thus is orthogonal to j, so there is a map $r^{\#}\colon X \to \mathrm{Path}\,\mathrm{Cyl}(j)$ filling in the square. One can check that the corresponding map $r\colon I \times X \to \mathrm{Cyl}(j)$ (defined by $r(t,x) = r^{\#}(t)(x)$) is a retraction, so that j is a cofibration.

Now suppose that $j \in {}^{\perp}\,\mathrm{fib}$. From the above, we know that j is a cofibration, and we need to show that it is also a homotopy equivalence. We first fill in the left hand diagram below to get a map $f\colon X \to W$ with $fj = 1_W$. We then define $c\colon W \to \mathrm{Path}(X)$ by $c(w)(t) = j(w)$, and apply Corollary 16 to the inclusion $\{0,1\} \rightarrowtail I$ to see that $(p_0,p_1)\colon \mathrm{Path}(E) \to E \times E$ is a fibration. This means that we can fill in the right hand square below to get a map $h^{\#}\colon X \to \mathrm{Path}(X)$ whose adjoint is a homotopy between 1_X and fj under W. This shows that j is a homotopy equivalence, as claimed. □