PURE MATHEMATICS

Spring Semester 2000 $2\frac{1}{2}$ **hours**

Rings, Modules and Linear Algebra

Answer ***four*** *questions. You are advised* ***not*** *to answer more than four questions: if you do, only your best four will be counted.*

1 You should justify your answers to parts (i) and (iii) of this question, but you need only state your answers to part (ii).

(i) Consider the following matrix over $\mathbb{C}$:

$$A = \begin{pmatrix} -1 & 1 & 1 & -1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

(a) What is the characteristic polynomial of A? ***(3 marks)***

(b) What is the minimal polynomial of A? ***(4 marks)***

(c) What is the rank of the matrix $A + I$? ***(3 marks)***

(ii) Consider a matrix B of the form $J(\lambda, k_1) \oplus \ldots \oplus J(\lambda, k_r)$, with $k_1 \leqslant \ldots \leqslant k_r$. (Here $J(\lambda, k)$ is the Jordan block of size k and eigenvalue λ.)

(a) What is the dimension of the $\mathbb{C}[x]$-module M_B as a vector space over $\mathbb{C}$? ***(1 mark)***

(b) What is the characteristic polynomial of B? ***(2 marks)***

(c) What is the minimal polynomial of B? ***(2 marks)***

(d) What is the rank of $B - \lambda I$? ***(2 marks)***

(iii) Which direct sum of basic $\mathbb{C}[x]$-modules is isomorphic to M_A (where A is as in part (i))? ***(8 marks)***

2 Consider the following matrix over $\mathbb{C}[x]$:

$$A = \begin{pmatrix} x & 0 & -1 \\ -1 & x & -1 \\ 0 & -1 & x+1 \end{pmatrix}.$$

Let N be the quotient of $\mathbb{C}[x]^3$ by the span of the columns of A.

(i) Reduce the matrix A to normal form by row and column operations. ***(10 marks)***

(ii) Give a polynomial $f(x)$ such that $N \simeq \mathbb{C}[x]/f(x)$. ***(2 marks)***

(iii) Give a matrix B such that $N \simeq M_B$. ***(3 marks)***

(iv) Give a list of basic $\mathbb{C}[x]$-modules whose direct sum is isomorphic to N. ***(4 marks)***

(v) Prove that the only $\mathbb{C}[x]$-module homomorphism from $\mathbb{C}[x]/x^3$ to N is zero. (If you use any theorem saying that certain homomorphisms are zero, then you should prove it.) ***(6 marks)***

3 (i) List all the isomorphism types of Abelian groups of order p^4, where p is prime. ***(5 marks)***

(ii) Let M be an Abelian group, p a prime number, and k a positive integer. Define the subgroups $F_p^k(M)$ and the integers $f_p^k(M)$, explaining carefully what your definition means. ***(6 marks)***

(iii) Calculate $f_p^1(\mathbb{Z}_{p^{k_1}} \oplus \ldots \oplus \mathbb{Z}_{p^{k_r}})$, where $0 < k_1 \leqslant \ldots \leqslant k_r$. ***(4 marks)***

(iv) If M is an Abelian group of order p^4 and $|\{m \in M \mid pm = 0\}| = p^3$, which of the groups in your list is isomorphic to M? ***(5 marks)***

(v) How many isomorphism classes are there of Abelian groups of order 10000? ***(5 marks)***

4 (i) State and prove the Chinese Remainder Theorem. ***(13 marks)***

(ii) Find a number e such that $e = 1 \pmod 7$ and $e = 0 \pmod 5$, and show that $\overline{e}^2 = \overline{e}$ in $\mathbb{Z}_{35}$. ***(4 marks)***

(iii) Let K be a module over a Euclidean domain R, and let L, M and N be submodules of K. Suppose that $aL = bM = cN = 0$ for some elements $a, b, c \in R$, and that $\gcd(a, b) = \gcd(a, c) = 1$. Prove that $L \cap (M + N) = \{0\}$. ***(8 marks)***

5 (i) State (without proof) the First Isomorphism Theorem for rings. ***(5 marks)***

(ii) Prove that the ring $\mathbb{R}[x]/(x^2 + 1)$ is isomorphic to $\mathbb{C}$. ***(9 marks)***

(iii) Let $\alpha: \mathbb{Z}[i] \longrightarrow \mathbb{Z}_5$ be defined by $\alpha(a + bi) = \overline{a - 2b}$.

(a) Show that α is a surjective ring homomorphism. ***(4 marks)***

(b) Let $q \in \mathbb{Z}[i]$ be such that $\ker(\alpha) = \mathbb{Z}[i]q$. Prove that $2 + i$ is divisible by q. ***(2 marks)***

(c) Prove that $\mathbb{Z}[i]/(2 + i) \simeq \mathbb{Z}_5$. (You may assume that $2 + i$ is an irreducible element of $\mathbb{Z}[i]$.) ***(5 marks)***

End of Question Paper