UNIVERSITY OF SHEFFIELD **PMA346**

PURE MATHEMATICS

Spring Semester 2000 $2\frac{1}{2}$ **hours**

Rings, Modules and Linear Algebra — Sample paper

Answer ***four*** *questions. You are advised* ***not*** *to answer more than four questions: if you do, only your best four will be counted.*

1 (i) Consider the following matrix over $\mathbb{C}$:

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

(a) What is the characteristic polynomial of A? ***(5 marks)***

(b) What is the minimal polynomial of A? ***(5 marks)***

(c) Which direct sum of basic $\mathbb{C}[x]$-modules is isomorphic to M_A? ***(5 marks)***

(ii) Now let a be a complex number and consider the matrix

$$B = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Determine the Jordan normal form of B, with careful attention to any special cases. ***(10 marks)***

2 Consider the following matrix over $\mathbb{Z}$:

$$A = \begin{pmatrix} 2 & 4 & 6 & 8 \\ 10 & 12 & 14 & 16 \\ 16 & 14 & 12 & 10 \\ 8 & 6 & 4 & 2 \end{pmatrix}$$

Let N be the quotient of $\mathbb{Z}^4$ by the span of the columns of A.

(i) Reduce the matrix A to normal form by row and column operations. ***(9 marks)***

(ii) Give a list of cyclic $\mathbb{Z}$-modules whose direct sum is isomorphic to N. ***(3 marks)***

(iii) Prove that N is neither a free module nor a torsion module. ***(6 marks)***

(iv) Prove that any homomorphism from $\mathbb{Z}_3$ to N is zero. ***(7 marks)***

3 (i) List all the isomorphism types of Abelian groups of order 32. ***(7 marks)***

(ii) State the value of $f_p^k(\mathbb{Z}_{q^j})$, where p and q are primes, and k and j are natural numbers. ***(5 marks)***

(iii) Let A be an Abelian group of order 32. Suppose that $4A = \{0\}$ and $f_2^2(A) = 2$. Which of the groups in your list is isomorphic to A? ***(8 marks)***

(iv) Let A' be an Abelian group of order 32 such that $g_2^1(A') = g_2^2(A') = 0$. Show that A' is cyclic. ***(5 marks)***

4 (i) Let R be a Euclidean domain. Prove that every ideal in R is principal. ***(7 marks)***

(ii) Put $I = \{f \in \mathbb{C}[x] \mid f(0) = f'(0) = f(1) = 0\}$; you may assume that this is an ideal. Find a polynomial $g(x)$ such that $I = \mathbb{C}[x]g(x)$ (and justify your answer). ***(6 marks)***

(iii) Put $M = \{(x, y, z) \in \mathbb{Z}^3 \mid x = y = z \pmod{7}\}$.

(a) By constructing a suitable homomorphism, show that $\mathbb{Z}^3/M \simeq \mathbb{Z}_7 \oplus \mathbb{Z}_7$. ***(6 marks)***

(b) Find a basis for M (and justify your answer). ***(6 marks)***

5 (i) Consider the following matrices:

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Find all the homomorphisms of $\mathbb{C}[x]$-modules from M_A to M_B. ***(7 marks)***

(ii) Let M and N be modules over a Euclidean domain R, and suppose there are coprime elements $a, b \in R$ such that $aM = bN = \{0\}$. Prove that any homomorphism from M to N is zero. ***(7 marks)***

(iii) Consider the following matrices:

$$C = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 10 \end{pmatrix} \qquad D = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 2 & 2 & 2 & 0 \\ 2 & 2 & 2 & 2 \end{pmatrix}$$

(a) Calculate $\mathrm{char}(C)$ and $\mathrm{char}(D)$. ***(4 marks)***

(b) Deduce that any homomorphism of $\mathbb{C}[x]$-modules from M_C to M_D is zero, explaining your reasoning carefully. ***(7 marks)***

End of Question Paper