Rings, Modules and Linear Algebra — Sample exam solutions

1. (i) (a) The characteristic polynomial is the determinant of the matrix

$$tI - A = \begin{pmatrix} t-1 & 0 & 0 & -1 \\ 0 & t+1 & -1 & 0 \\ 0 & -1 & t+1 & 0 \\ -1 & 0 & 0 & t-1 \end{pmatrix}.$$

If we add $t-1$ times the 4'th column to the first column, and $t+1$ times the third column to the second column, this becomes

$$\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & t^2+2t & t+1 & 0 \\ t^2-2t & 0 & 0 & t-1 \end{pmatrix}.$$

From this we see that $\operatorname{char}(A) = (-1)^2(t^2+2t)(t^2-2t) = t^2(t-2)(t+2)$.

(b) The minimal polynomial is a divisor of the characteristic polynomial and has the same roots, so it must be either $t(t-2)(t+2) = t^3 - 4t$ or $t^2(t-2)(t+2) = t^4 - 4t^2$. We find that

$$A^2 = \begin{pmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & -2 & 0 \\ 0 & -2 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 4 & 0 & 0 & 4 \\ 0 & -4 & 4 & 0 \\ 0 & 4 & -4 & 0 \\ 4 & 0 & 0 & 4 \end{pmatrix} = 4A.$$

Thus $A^3 - 4A = 0$ and we must have $\min(A) = t^3 - 4t$.

(c) The factor $t-2$ in $\operatorname{char}(A)$ comes from a Jordan block $J(2,1)$, and the factor $t+2$ comes from $J(-2,1)$. The factor t^2 comes from $J(0,2)$ or $J(0,1) \oplus J(0,1)$. However, if we had a $J(0,2)$ then we would have a factor t^2 rather than t in $\min(A)$. Thus, the Jordan Normal Form of A must be $J(2,1) \oplus J(-2,1) \oplus J(0,1) \oplus J(0,1)$, so

$$M_A \simeq B(2,1) \oplus B(-2,1) \oplus B(0,1) \oplus B(0,1).$$

(ii) We have

$$\operatorname{char}(B)(t) = \det \begin{pmatrix} t & -a & 0 \\ 0 & t & -1 \\ 0 & 0 & t \end{pmatrix} = t^3,$$

so the Jordan normal form of B consists of blocks $J(0,1)$ or $J(0,2)$ or $J(0,3)$.
We also see that $\min(B)(t)$ is t or t^2 or t^3. We have

$$B \neq 0$$

$$B^2 = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$B^3 = 0.$$

If $a \neq 0$ we see that $\min(B)(t) = t^3$, so we must have a block $J(0,3)$. As this is a 3×3 matrix it must be the whole of the JNF of B.
If $a = 0$ we see that $\min(B)(t) = t^2$, so we must have a block $J(0,2)$ but not a block $J(0,3)$. The only way we can make this up to a 3×3 matrix is for the JNF to be $J(0,2) \oplus J(0,1)$.

2. (i)

$$\begin{pmatrix} 2 & 4 & 6 & 8 \\ 10 & 12 & 14 & 16 \\ 16 & 14 & 12 & 10 \\ 8 & 6 & 4 & 2 \end{pmatrix} \to \begin{pmatrix} 2 & 0 & 0 & 0 \\ 10 & -8 & -16 & -24 \\ 16 & -18 & -36 & -54 \\ 8 & -10 & -20 & -30 \end{pmatrix} \to \begin{pmatrix} 2 & 0 & 0 & 0 \\ 10 & 8 & 0 & 0 \\ 16 & 18 & 0 & 0 \\ 8 & 10 & 0 & 0 \end{pmatrix}$$

$$\to \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(ii) It follows that

$$N \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z} \oplus \mathbb{Z}.$$

(iii) Put $u = (1,0,0,0)$ and $v = (0,0,1,0)$, considered as elements of $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z} \oplus \mathbb{Z} \simeq N$. Then $u \neq 0$ but $2u = 0$, so N is not torsion-free and so is not free. On the other hand, we have $mv = (0,0,m,0) \neq 0$ for all $m \neq 0$, so v is not a torsion element and N is not a torsion module.

(iv) Let $\alpha\colon \mathbb{Z}_3 \to N$ be a homomorphism. Then $\alpha(\overline{1}) = (a,b,c,d)$ for some $a,b \in \mathbb{Z}_2$ and $c,d \in \mathbb{Z}$. We then have

$$(3a, 3b, 3c, 3d) = 3\alpha(\overline{1}) = \alpha(\overline{3}) = \alpha(\overline{0}) = (0,0,0,0).$$

As $a,b \in \mathbb{Z}_2$ we have $a = 3a$ and $b = 3b$. As $3a = 3b = 0$ we conclude that $a = b = 0$. Also, c and d are just integers and $3c = 3d = 0$ so $c = d = 0$. Thus $\alpha(\overline{1}) = (0,0,0,0)$ and so $\alpha(\overline{k}) = k\alpha(\overline{1}) = (0,0,0,0)$ for all k, so $\alpha = 0$.

3. (i) Any Abelian group A of order $32 = 2^5$ can be written as a direct sum of groups of the form $\mathbb{Z}_{2^k}$ with $1 \leq k \leq 5$. If $A \simeq \mathbb{Z}_{2^{k_1}} \oplus \dots \mathbb{Z}_{2^{k_r}}$ then

$$2^5 = |A| = 2^{k_1} \times \dots \times 2^{k_r} = 2^{k_1 + \dots + k_r},$$

so $k_1 + \dots + k_r = 5$. As each k_i is at least 1 this means that $r \leq 5$. Using this and some trial and error we see that the possibilities are as follows:

$$\begin{aligned} A_1 &= \mathbb{Z}_{32} \\ A_2 &= \mathbb{Z}_{16} \oplus \mathbb{Z}_2 \\ A_3 &= \mathbb{Z}_8 \oplus \mathbb{Z}_4 \\ A_4 &= \mathbb{Z}_8 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \\ A_5 &= \mathbb{Z}_4 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2 \\ A_6 &= \mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \\ A_7 &= \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2. \end{aligned}$$

(ii) We have $\mathbb{Z}_{q^j} = B_q^j$ so

$$f_p^k(\mathbb{Z}_{q^j}) = \begin{cases} 0 & \text{if } p \neq q \\ 0 & \text{if } p = q \text{ and } j < k \\ 1 & \text{if } p = q \text{ and } j \geq k. \end{cases}$$

(iii) We see from the above that $f_2^2(\mathbb{Z}_2) = 0$ and $f_2^2(\mathbb{Z}_4) = f_2^2(\mathbb{Z}_8) = f_2^2(\mathbb{Z}_{16}) = f_2^2(\mathbb{Z}_{32}) = 1$, so

$$\begin{aligned} f_2^2(A_1) &= 1 \\ f_2^2(A_2) &= 1 \\ f_2^2(A_3) &= 2 \\ f_2^2(A_4) &= 1 \\ f_2^2(A_5) &= 2 \\ f_2^2(A_6) &= 1 \\ f_2^2(A_7) &= 0. \end{aligned}$$

Thus, as $f_2^2(A) = 2$ we must have $A \simeq A_3$ or $A \simeq A_5$. However, we also have $4A = \{0\}$ ad $4A_3 \neq \{0\}$ because A_3 contains $\mathbb{Z}_8$. We must therefore have $A \simeq A_5 = \mathbb{Z}_4 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2$.

4. (i) Let I be an ideal; we must find an element $b \in R$ such that $I = Rb$. First, if $I = \{0\}$ then $I = R0$ as required; so we may assume that $I \neq \{0\}$. Each nonzero element $b \in I$ has a valuation $\nu(b) \in \mathbb{Z}$ with $\nu(b) \geq 0$. Choose such an element for which $\nu(b)$ is as small as possible. By assumption $b \in I$ and I is an ideal so $Rb \subseteq I$.
Conversely, suppose that $a \in I$. By axiom (b), then there are elements $q, r \in R$ such that $a = bq + r$ and either $r = 0$ or $\nu(r) < \nu(b)$. Note that $r = a - bq$ and $a, b \in I$ so $r \in I$. If r were a nonzero element of I with $\nu(r) < \nu(b)$, this would contradict our choice of b. We must therefore have $r = 0$ instead, so $a = qb$, so $a \in Rb$. This proves that $I \subseteq Rb$ and thus that $I = Rb$.

(ii) If $f(x) = \sum_i a_i x^i$ then $f(0) = a_0$ and $f'(0) = a_1$. Thus if $f \in I$ we must have $a_0 = a_1 = 0$, so f is divisible by x^2. We also have $f(1) = 0$ so f is divisible by $(x-1)$. As x^2 and $x - 1$ are coprime it follows that f is divisible by $x^2(x-1)$.
Conversely, if $f(x) = h(x)x^2(x-1) = h(x)(x^3 - x^2)$ for some h then clearly $f(0) = f(1) = 0$. Moreover, we have $f'(x) = h'(x)(x^3 - x^2) + h(x)(3x^2 - 2x)$ and so $f'(0) = 0$, so $f \in I$. Thus $I = \mathbb{C}[x].x^2(x-1)$.

(iii) (a) Define a homomorphism $\alpha\colon \mathbb{Z}^3 \to \mathbb{Z}_7 \oplus \mathbb{Z}_7$ by $\alpha(x, y, z) = (\overline{x-y}, \overline{y-z})$. Clearly $\alpha(x, y, z) = (\overline{0}, \overline{0})$ iff $x = y = z \pmod 7$ iff $(x, y, z) \in M$, so $M = \ker(\alpha)$. Moreover, given any point $(\overline{a}, \overline{b}) \in \mathbb{Z}_7 \oplus \mathbb{Z}_7$ we have $\alpha(a, 0, -b) = (\overline{a}, \overline{b})$, so α is surjective. It now follows from the First Isomorphism Theorem that $\mathbb{Z}_7 \oplus \mathbb{Z}_7 \simeq \mathbb{Z}^3/M$.

(b) Put $u_1 = (1, 1, 1)$ and $u_2 = (0, 7, 0)$ and $u_3 = (0, 0, 7)$; these are clearly linearly independent elements of M. If $(x, y, z) \in M$ we have $x - y = 7a$ and $y - z = 7b$ for some $a, b \in \mathbb{Z}$, so $y = x - 7a$ and $z = y - 7b = x - 7a - 7b$, so

$$(x, y, z) = (x, x - 7a, x - 7a - 7b) = xu_1 - au_2 - (a + b)u_3.$$

This shows that $\{u_1, u_2, u_3\}$ generates M, so it is a basis.

5. (i) The $\mathbb{C}[x]$-module homomorphisms from M_A to M_B correspond to 2×4 matrices C such that $CA = BC$. If

$$C = \begin{pmatrix} a & b \\ c & d \\ e & f \\ g & h \end{pmatrix}$$

then we have

$$CA = \begin{pmatrix} -b & a \\ -d & c \\ -f & e \\ -h & g \end{pmatrix} \qquad BC = \begin{pmatrix} g & h \\ a & b \\ c & d \\ e & f \end{pmatrix}.$$

By comparing the top rows we see that $h = a$ and $g = -b$. By comparing the second rows we see that $d = -a$ and $c = b$. By comparing the third rows we see that $f = -c = -b$ and $e = d = -a$. Given all this, we see that $e = -a = -h$ and $f = -b = g$, so the bottom rows are automatically equal. We thus see that the homomorphisms are precisely the matrices of the form

$$C = \begin{pmatrix} a & b \\ b & -a \\ -a & -b \\ -b & a \end{pmatrix}$$

for some $a, b \in \mathbb{C}$.

(ii) As a and b are coprime, we can choose $u, v \in R$ such that $au + bv = 1$. Let $\alpha\colon M \to N$ be a homomorphism. If $m \in M$ then $am = 0$ (because $aM = \{0\}$) so $a\alpha(m) = \alpha(am) = \alpha(0) = 0$. We also have $\alpha(m) \in N$ and $bN = \{0\}$ so $b\alpha(m) = 0$. It follows that $\alpha(m) = 1.\alpha(m) = ua\alpha(m) + vb\alpha(m) = 0 + 0 = 0$. This holds for all $m \in M$ so $\alpha = 0$ as required.

(iii) (a)

$$\mathrm{char}(C)(t) = \det \begin{pmatrix} t-1 & -2 & -3 & -4 \\ 0 & t-5 & -6 & -7 \\ 0 & 0 & t-8 & -9 \\ 0 & 0 & 0 & t-10 \end{pmatrix} = (t-1)(t-5)(t-8)(t-10)$$

$$\mathrm{char}(D)(t) = \det \begin{pmatrix} t-2 & 0 & 0 & 0 \\ -2 & t-2 & 0 & 0 \\ -2 & -2 & t-2 & 0 \\ -2 & -2 & -2 & t-2 \end{pmatrix} = (t-2)^4.$$

(b) Put $f(x) = \mathrm{char}(C)(x) = (x-1)(x-5)(x-8)(x-10)$. The Cayley-Hamilton theorem tells us that $f(C) = 0$. Thus, for $m \in M_C$ we have $f(x).m = f(C)m = 0$. Similarly, if we put $g(x) = (x-2)^4$ then $g(x).n = 0$ for all $n \in M_D$. As f and g are coprime, part (ii) tells us that the only homomorphism from M_C to M_D is zero.