UNIVERSITY OF SHEFFIELD **PMA346**

PURE MATHEMATICS

Spring Semester 2005 2 **hours** 30 **minutes**

Rings, Modules and Linear Algebra

Answer ***four*** *questions. You are advised* ***not*** *to answer more than four questions: if you do, only your best four will be counted.*

1 (i) Consider the following matrix over $\mathbb{C}$:

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

(a) What is the characteristic polynomial of A? ***(4 marks)***

(b) What are the ranks of A and $A - 2I$? ***(4 marks)***

(c) Which direct sum of basic $\mathbb{C}[x]$-modules is isomorphic to M_A? ***(4 marks)***

(ii) Let M be an abelian group of order 1296, and suppose that $18m = 0$ for all $m \in M$. What are the possible isomorphism types for M? ***(7 marks)***

(iii) Consider the matrix

$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

Show that for any polynomial $f(x) = a_0 + a_1 x + \cdots + a_n x^n$, we have

$$f(A) = \begin{pmatrix} f(\lambda) & f'(\lambda) \\ 0 & f(\lambda) \end{pmatrix}.$$

(6 marks)

2 (i) Let N be the subgroup of $\mathbb{Z}^3$ generated by the columns of the following matrix:

$$B = \begin{pmatrix} 200 & 100 & 108 \\ 36 & 0 & 72 \\ 0 & 0 & 36 \end{pmatrix}$$

Reduce the matrix to normal form, and thus write $\mathbb{Z}^3/N$ as a direct sum of cyclic groups. ***(8 marks)***

(ii) Let R be a Euclidean domain, and let a, b, c, d be elements of R. Let M be the submodule of R^4 generated by the columns of the following matrix:

$$A = \begin{pmatrix} a & 1 & 0 & 0 \\ 0 & b & 1 & 0 \\ 0 & 0 & c & 1 \\ 0 & 0 & 0 & d \end{pmatrix}$$

Show that R^4/M is cyclic, and find $x \in R$ such that $R^4/M \simeq R/x$. ***(8 marks)***

(iii) Let L be the subgroup of $\mathbb{Z}^3$ generated by the vectors $u_1 = (1, 3, 6)$, $u_2 = (0, 2, 6)$ and $u_3 = (0, 0, 4)$. Find a basis $\{v_1, v_2, v_3\}$ for $\mathbb{Z}^3$ and integers d_1, d_2, d_3 such that $\{d_1v_1, d_2v_2, d_3v_3\}$ is a basis for L over $\mathbb{Z}$. Only brief justification is required. ***(9 marks)***

3 (i) Let R be a Euclidean domain, let M be a finitely generated R-module, and let N be a submodule of M. Prove that N is also finitely generated. (You may assume that every submodule of a finitely generated free module is again a finitely generated free module.) ***(12 marks)***

(ii) Consider the following matrices over $\mathbb{C}$:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 & 3 \\ 3 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix}$$

We use these to construct modules M_A and M_B over $\mathbb{C}[x]$ in the usual way.

(a) Show that $(x^3 - 27)v = 0$ for all $v \in M_B$, and that $x(x-3)u = 0$ for all $u \in M_A$. Deduce that $(x^3 - 9x)u = 0$ for all $u \in M_A$. ***(4 marks)***

(b) Let $\gamma\colon M_A \longrightarrow M_B$ be a homomorphism, given by $\gamma(u) = Cu$ for some 3×3 matrix C over $\mathbb{C}$. Show that $(x - 3)\gamma(u) = 0$ for all $u \in M_A$. ***(5 marks)***

(c) Put $V = \{(x, x, x) \mid x \in \mathbb{C}\} \subseteq M_B = \mathbb{C}^3$. Show that the image of γ is contained in V. ***(4 marks)***

4 (i) Let R be a Euclidean domain, and let a and b be two coprime elements of R. Let M and N be R-modules such that $aM = \{0\}$ and $bN = \{0\}$. Show that the only R-module homomorphism from M to N is zero. ***(6 marks)***

(ii) Let L and M be finite abelian groups, and let N be a subgroup of M such that $|L|$ is coprime to $|M/N|$.

(a) Show that the only homomorphism from L to M/N is zero. ***(2 marks)***

(b) By considering the usual homomorphism $\pi\colon M \longrightarrow M/N$, deduce that for any homomorphism $\alpha\colon L \longrightarrow M$, we have $\alpha(L) \subseteq N$. ***(4 marks)***

(iii) (a) Put $U = \{f(t) \in C^{\infty}(\mathbb{R}, \mathbb{R}) \mid f(0) = f(1) = f(-1) = 0\}$. Is this an $\mathbb{R}[D]$-submodule of $C^{\infty}(\mathbb{R}, \mathbb{R})$? Justify your answer. ***(3 marks)***

(b) Let W_d denote the set of polynomials $f(t) \in \mathbb{R}[t]$ of degree less than or equal to d, considered as an $\mathbb{R}[D]$-module in the usual way. Show that W_d is cyclic. ***(4 marks)***

(c) Let V be the set of functions of the form $p\sin(t) + q\cos(t)$ with $p, q \in \mathbb{R}$. Show that the only $\mathbb{R}[D]$-module homomorphism from W_d to V is zero. ***(6 marks)***

5 Let R be a Euclidean domain, and let a and b be nonzero elements of R. Define a map $\phi\colon R/ab \longrightarrow R/a \times R/b$ by $\phi(u + Rab) = (u + Ra, u + Rb)$; you may assume that this is a well-defined ring homomorphism.

(i) Prove that if a and b are coprime, then ϕ is an isomorphism (the Chinese Remainder Theorem). ***(10 marks)***

(ii) Conversely, suppose that ϕ is surjective, so that in particular the element $(1 + Ra, 0 + Rb)$ lies in the image of ϕ. Prove that a and b are coprime. ***(6 marks)***

(iii) Prove that for any $\lambda \in \mathbb{C}$, the ring $\mathbb{C}[x]/(x - \lambda)$ is isomorphic to $\mathbb{C}$. ***(4 marks)***

(iv) Find an ideal $I \leqslant \mathbb{C}[x]$ such that $\mathbb{C}[x]/I$ is isomorphic to $\mathbb{C} \times \mathbb{C} \times \mathbb{C} \times \mathbb{C}$ as rings. ***(5 marks)***

End of Question Paper