Rings and Modules Problems

1. Rings and fields

Q1: Which of the following are commutative rings?

- R_0 is the set of polynomials $f(x) \in \mathbb{R}[x]$ such that $f(-x) = f(x)$.
- R_1 is the set of polynomials $f(x) \in \mathbb{R}[x]$ such that $f(-x) = -f(x)$.
- R_2 is the set of 2×2 matrices over $\mathbb{R}$, with the usual definition of matrix multiplication.
- R_3 is the set of 2×2 matrices over $\mathbb{R}$, with multiplication given by the definition

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} aa' & bb' \\ cc' & dd' \end{pmatrix}.$$

- R_4 is the set of vectors in $\mathbb{R}^3$, with multiplication given by the cross product:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \times \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} yz'-y'z \\ zx'-z'x \\ xy'-x'y \end{pmatrix}.$$

Q2: Choose two typical elements a and b of the ring $\mathbb{Z}_{(5)}$. Find $a+b$ and ab and check that they lie in $\mathbb{Z}_{(5)}$. Repeat this for the rings $\mathbb{Z}[i]$, $\mathbb{Q}[x,y]$ and $\mathbb{Z}_{12}$.

Q3: Let R be the set of rational numbers that can be written in the form a/b, where b is not divisible by 6. (We would call this $\mathbb{Z}_{(6)}$ if 6 were prime, which of course it isn't). Prove that R is *not* a subring of $\mathbb{Q}$.

Q4: Let K be a field; prove that K is an integral domain.

Q5: Let X be a set. Let R be the set of all subsets of X, and define addition and multiplication of subsets as follows.

$$\begin{aligned} A + B &= (A \cup B) \setminus (A \cap B) \\ &= \{x \in X \mid x \in A \text{ or } x \in B \text{ but not both. }\} \\ AB &= A \cap B. \end{aligned}$$

For any $A \in R$, we define a function $\chi_A \colon X \to \mathbb{Z}_2$ by

$$\chi_A(x) = \begin{cases} \overline{1} & \text{if } x \in A \\ \overline{0} & \text{if } x \notin A \end{cases}$$

(a) Check that $A + \emptyset = A$ and $A + A = \emptyset$.
(b) Show that $\chi_{A+B}(x) = \chi_A(x) + \chi_B(x)$ and $\chi_{AB}(x) = \chi_A(x)\chi_B(x)$.
(c) Show that if $\chi_A = \chi_B$ then $A = B$.
(d) Prove that the definitions above make R into a commutative ring. (You may wish to use (b) and (c) to help check some of the axioms.)

2. Modules

Q6: List all the elements of the Abelian group $\mathbb{Z}_2 \oplus \mathbb{Z}_5$. Find an element that has order 10.

Q7:

(a) Calculate $(1 + D + D^2/2 + D^3/6).t^3$. What do you notice? Can you guess a generalisation?
(b) Put $f(t) = e^{-t}\sin(t)$ so $f \in C^\infty(\mathbb{R}, \mathbb{R})$. Calculate $(D+1)^2 f$.
(c) Put $g_k(t) = t^k e^t$, so $g_k \in C^\infty(\mathbb{R}, \mathbb{R})$. Calculate $(D-1)g_k$ and thus $(D-1)^k g_k$. (You may wish to try $k = 3$ first.)

(d) Put $f(t) = te^t$. Show that $(D^k f)(t) = (k+t)e^t$ for all $k \geq 0$ and thus that $(p(D)f)(t) = (p'(1) + p(1)t)e^t$.

Q8: Define $v(t) = e^{t^2/2}$, so $u \in C^\infty(\mathbb{R}, \mathbb{R})$. Let V be the set of functions of the form $f(t)v(t)$, where f is a polynomial. For example, the function $(1+t+t^2)e^{t^2/2}$ is an element of V.

(a) Prove that V is an $\mathbb{R}[D]$-submodule of $C^\infty(\mathbb{R}, \mathbb{R})$.
(b) Calculate $D^k v$ for $0 \leq k \leq 3$.
(c) Show that for all $k \geq 0$ there is a polynomial $p_k(t)$ of the form $t^k +$ lower terms such that $D^k v = p_k.v$.
(d) Show that if $q(D)$ is a nonzero element of $\mathbb{R}[D]$ then $q(D)v \neq 0$ (look at leading terms).
(e) Suppose that $f(t)$ is a polynomial of degree k, say $f(t) = at^k +$ lower terms . Prove by induction on k that $fv = q(D)v$ for some element $q(D) \in \mathbb{R}[D]$.
(f) Deduce that $V \simeq \mathbb{R}[D]$ as an $\mathbb{R}[D]$-module.

3. Modules over polynomial rings

Q9: Let A be the matrix $\left(\begin{smallmatrix} 1&1&0&0\\ 0&1&1&0\\ 0&0&1&1\\ 0&0&0&1 \end{smallmatrix}\right)$. Find A^2, A^3 and A^4. Can you give a general rule for A^n?

Q10:

(a) Put $A = \left(\begin{smallmatrix} 1&0&1\\ 0&1&0\\ 0&0&1 \end{smallmatrix}\right)$ and $m = (1,1,1) \in M_A$. Calculate $(x^3-1)m$.
(b) Put $A = \left(\begin{smallmatrix} 0&0&1\\ 0&1&0\\ 1&0&0 \end{smallmatrix}\right)$ and $m = (1,2,3) \in M_A$. Calculate $(x^3-1)m$.
(c) Put $A = \left(\begin{smallmatrix} 1&1\\ 1&1 \end{smallmatrix}\right)$ and $m = (1,-1) \in M_A$. Calculate $(14x^{12} + 5x^{11} - 36x^7 - 22x^4 + 13x - 5)m$. (You may wish to start by calculating fm for some very simple polynomials f first.)

Q11: Let $A = \left(\begin{smallmatrix} a&b\\ c&d \end{smallmatrix}\right)$ be a 2×2 matrix, and put $f(x) = x^2 - (a+d)x + (ad-bc)$. Show that $f(A) = \left(\begin{smallmatrix} 0&0\\ 0&0 \end{smallmatrix}\right)$. (This is the 2×2 case of the Cayley-Hamilton theorem.)

Q12: Put $A = \left(\begin{smallmatrix} 1&1\\ 1&1 \end{smallmatrix}\right)$ and $f(x) = x^4 - 3x$. Calculate $f(A)$.

Q13: Consider the matrix

$$A = \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right),$$

and a polynomial $f(x) = \sum_i a_i x^i$.

(a) Calculate A^i for some small numbers i, then give the general rule.
(b) Write $b = a_0 + a_2 + a_4 + \ldots = \sum_j a_{2j}$ and $c = a_1 + a_3 + \ldots = \sum_j a_{2j+1}$. Express $f(1)$ and $f(-1)$ in terms of b and c.
(c) Show that

$$f(A) = \frac{f(1)}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \frac{f(-1)}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

4. General module theory

Q14: Let $\alpha \colon M \to N$ be a homomorphism of modules over $\mathbb{C}[x]$. Suppose that $(x^3 - x)M = \{0\}$ and $x^5 N = \{0\}$. Prove that for $n \in \text{image}(\alpha)$ we have $xn = 0$. Can you formulate a general theorem of which this is a special case?

Q15: Let R be a ring, and let M and N be R-modules. Show that if $M \oplus N$ is cyclic, then so are M and N.

Q16:

(a) Put $N_0 = \{(n, m) \in \mathbb{Z}^2 \mid n - m \text{ is even }\}$ and $N_1 = \{(n, m) \in \mathbb{Z}^2 \mid n - m \text{ is odd }\}$. Are these $\mathbb{Z}$-submodules of $\mathbb{Z}^2$?

(b) Put $A = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ and $N_0 = \{(u, v) \in \mathbb{R}^2 \mid u - v = 0\}$ and $N_1 = \{(u, v) \mid u + v = 0\}$. Are these $\mathbb{R}[x]$-submodules of M_A?

(c) Put $N_0 = \{f \in C^\infty(\mathbb{R}, \mathbb{R}) \mid f(1) = 0\}$ and $N_1 = \{f \in C^\infty(\mathbb{R}, \mathbb{R}) \mid \int_0^2 f = 0\}$. Are these $\mathbb{R}[D]$-submodules of $C^\infty(\mathbb{R}, \mathbb{R})$?

Q17: Let R be a ring with exactly 10 elements, and let M be an R-module with exactly 20 elements. Prove that M is not a free module.

Q18: For any integer d, let N_d be the submodule of $\mathbb{Z}_{24}$ generated by $\overline{d}$. The group N_6 has precisely 4 elements; list them. Find integers d and e such that $N_6 \cap N_4 = N_d$ and $N_6 + N_4 = N_e$.

Q19: For any natural number d dividing 900, let N_d be the submodule of $\mathbb{Z}_{900}$ generated by $\overline{d}$.

(a) What is the order of N_{10}?

(b) Which standard group is isomorphic to $\mathbb{Z}_{900}/N_{10}$?

(c) Find d such that the submodule generated by $\overline{70}$ is N_d.

(d) Find d such that $N_{12} + N_{30} + N_{100} = N_d$.

(e) Find d such that $N_{30} \cap N_{50} = N_d$.

Q20: Consider the following matrix over $\mathbb{Q}$.

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Show that the module M_A over $\mathbb{Q}[x]$ is cyclic, and give a polynomial $f(x)$ such that $M_A \simeq \mathbb{Q}[x]/f(x)$.

Q21: Let W_d be the set of polynomials $f(t)$ of degree at most d. Prove that W_d is a cyclic module over $\mathbb{R}[D]$. What is the ideal $I \subseteq \mathbb{R}[D]$ such that $W_d \simeq \mathbb{R}[D]/I$?

Q22: Fix a nonzero vector $u \in \mathbb{R}^3$, and write $r = \|u\|$. Define an endomorphism $\phi\colon \mathbb{R}^3 \to \mathbb{R}^3$ by $\phi(v) = u \times v$. Write M for $\mathbb{R}^3$, considered as a module over $\mathbb{R}[x]$ using ϕ. Let L be the line through u and 0, and let K be the plane perpendicular to L.

(a) Show that L is a submodule of M, and that $xL = 0$.

(b) Show that K is a submodule of M, and that $(x^2 + r^2)K = 0$ and $xM \leq K$.

(c) Show that $(x^3 + r^2 x)M = 0$.

(You will need a number of standard facts about dot and cross products of vectors.)

5. Homomorphisms

Q23: In each of the following situations, find all the $\mathbb{Q}[x]$-module homomorphisms from M_A to M_B.

(a) $A = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

(b) $A = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} \mu & 0 \\ 0 & \lambda \end{smallmatrix}\right)$, where $\lambda \neq \mu$.

(c) $A = I_2$ (the 2×2 identity matrix) and $B = I_3$.

Q24:

(a) Put $A = \left(\begin{smallmatrix} 1 & 2 \\ 0 & 2 \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. Find all the $\mathbb{Q}[x]$-module homomorphisms from M_A to M_B.

(b) Put $A = \left(\begin{smallmatrix}1&0&0\\1&1&0\\0&1&1\end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix}1&0&0\\0&1&0\\0&0&1\end{smallmatrix}\right)$. Find all the $\mathbb{Q}[x]$-module homomorphisms from M_A to M_B.

(c) Suppose that $\lambda_1, \lambda_2, \lambda_3, \mu_1, \mu_2 \in \mathbb{C}$, and put $A = \begin{pmatrix}\lambda_1&0&0\\0&\lambda_2&0\\0&0&\lambda_3\end{pmatrix}$ and $B = \left(\begin{smallmatrix}\mu_1&0\\0&\mu_2\end{smallmatrix}\right)$. Show that for most values of the λ's and μ's, the only $\mathbb{C}[x]$-module homomorphism from M_A to M_B is zero. What can you say about the exceptional cases?

Q25: Let a be an element of a ring R. For any R-module M, put

$$\operatorname{ann}(a, M) = \{m \in M \mid am = 0\}.$$

We will also write R/a for the factor module R/Ra.

(a) Find $\operatorname{ann}(4, \mathbb{Z}_{12})$.

(b) Find $\operatorname{ann}(x-1, M_A)$, where A is the matrix $\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$.

(c) Let $\alpha\colon R/a \to M$ be a homomorphism. Show that $\alpha(\overline{1}) \in \operatorname{ann}(a, M)$.

(d) Conversely, given $m \in \operatorname{ann}(a, M)$, show that there is a unique homomorphism $\alpha\colon R/a \to M$ such that $\alpha(\overline{1}) = m$.

(e) Describe all the $\mathbb{R}[D]$-module homomorphisms from $\mathbb{R}[D]/(D^2-1)$ to $C^\infty(\mathbb{R}, \mathbb{R})$.

Q26: Show that there are homomorphisms $\alpha\colon \mathbb{Z}_3 \to \mathbb{Z}_9$ and $\beta\colon \mathbb{Z}_9 \to \mathbb{Z}_3$ given by $\alpha(\overline{n}) = \overline{3n}$ and $\beta(\overline{m}) = \overline{m}$. Show that the sequence $\mathbb{Z}_3 \xrightarrow{\alpha} \mathbb{Z}_9 \xrightarrow{\beta} \mathbb{Z}_3$ is exact.

Q27:

(a) Let α be a $\mathbb{C}[x]$-module homomorphism from $\mathbb{C}[x]/(x^2-1)$ to M_A, where $A = \left(\begin{smallmatrix}1&1&1\\1&1&1\\1&1&1\end{smallmatrix}\right)$. Prove that $\alpha = 0$.

(b) Let V be the space of functions of the form $a\sin(t) + b\cos(t)$, and let W be the space of functions of the form $a\sinh(t) + b\cosh(t)$. Let $\beta\colon V \to W$ be an $\mathbb{R}[D]$-module homomorphism. Show that $\beta = 0$.

(c) Let $\gamma\colon \mathbb{Z}_4 \to \mathbb{Z}^4$ be a homomorphism of $\mathbb{Z}$-modules. Prove that $\gamma = 0$.

Q28: Let R be a ring, and let M be an R-module with only finitely many elements, say $|M| = m$. How many homomorphisms are there from R^d to M?

6. Factor modules

Q29: Let M be a module over a ring R, and let L and N be submodules of M. Prove that $L/(L \cap N)$ is isomorphic to $(L+N)/N$. [You may wish to consider the homomorphism $\pi\colon L \to (L+N)/N$ given by $\pi(x) = x + N$.]

Q30: Let M be a module over a ring R, and let N_0 and N_1 be submodules of M. Define a homomorphism $\sigma\colon N_0 \oplus N_1 \to M$ by $\sigma(n_0, n_1) = n_0 + n_1$. Prove that σ is an isomorphism if and only if M is the internal direct sum of N_0 and N_1.

7. Ideals and factor rings

Q31:

(a) Show that there are no ring homomorphisms from $\mathbb{Z}_3$ to $\mathbb{Z}$. [Consider the equation $\overline{1}+\overline{1}+\overline{1} = \overline{0}$].

(b) Show that there are no ring homomorphisms from $\mathbb{Q}$ to $\mathbb{Z}$. [Consider the equation $\frac{1}{2}.(1+1) = 1$.]

(c) Show that there are no ring homomorphisms from $\mathbb{C}$ to $\mathbb{R}$.

(d) Find a ring homomorphism from $\mathbb{C}$ to $\mathbb{C}$ that is not the identity (there is only one reasonable example).

Q32:

(a) Prove that $\mathbb{Q}[x]/(x^2-2)$ is isomorphic to a subring of $\mathbb{R}$.
(b) Let I be the ideal $\mathbb{Z}[i].(2+3i)$ in $\mathbb{Z}[i]$, and define a homomorphism $\alpha\colon \mathbb{Z} \to \mathbb{Z}[i]/I$ by $\alpha(n) = n + I$.
  (i) Show that $\alpha(-5) = i + I$, and deduce that α is surjective.
  (ii) Suppose that $n \in \mathbb{Z}$ and that n is divisible by $2+3i$ in $\mathbb{Z}[i]$. Show that n^2 is divisible by 13 in $\mathbb{Z}$.
  (iii) Show that $\mathbb{Z}[i]/I \simeq \mathbb{Z}_{13}$.

Q33:

(a) Prove that the ring $\mathbb{R}[x]/(x^2+4)$ is isomorphic to $\mathbb{C}$.
(b) Prove that $\mathbb{R}[x]/(x^2-4)$ is not a field (and thus cannot be isomorphic to $\mathbb{C}$).

Q34: Let R be the ring $\mathbb{Z}[i]/3$, and put $u = 1 + i + 3\mathbb{Z}[i] \in R$.

(a) List the elements of R.
(b) Calculate u^k for $0 \le k \le 8$.
(c) Compare your list in (a) with your list in (b), and show that R is a field.
(d) Do you know another proof that R is a field?

8. Euclidean domains

Q35: Let n and m be coprime positive integers, and put $f(x) = x^n - 1$ and $g(x) = x^m - 1$. By considering the roots of f and g, show that the gcd of f and g in $\mathbb{C}[x]$ is $x-1$.

Q36: Let p be a prime, and let a and b be nonzero elements of $\mathbb{Z}_{(p)}$. Show that either a is a gcd of a and b in $\mathbb{Z}_{(p)}$, or b is a gcd of a and b in $\mathbb{Z}_{(p)}$.

9. Factorisation in Euclidean domains

10. Finite free modules over a Euclidean domain

Q37: Find bases over $\mathbb{Z}$ for the following submodules of $\mathbb{Z}^3$. Justify your answers.

(a) $M_0 = \{(x,y,z) \mid x - y + z = 0 \pmod 5\}$
(b) $M_1 = \{(x,y,z) \mid x = y \pmod 2 \text{ and } y = z \pmod 3\}$
(c) $M_2 = \{(x,y,z) \mid 6x + 15y + 10z = 0\}$.

Q38: Let $d_1, \dots, d_n$ be elements of a ring R, and let N be the submodule of R^n generated by the elements $d_1e_1, \dots, d_ne_n$. Prove that $R^n/N \simeq R/d_1 \oplus \dots \oplus R/d_n$. [You may wish to start by defining a homomorphism $\alpha\colon R^n \to R/d_1 \oplus \dots \oplus R/d_n$.]

Q39: Put

$$F = \{(w,x,y,z) \in \mathbb{Z}^4 \mid w + x + y + z \text{ is even }\}$$
$$G = \{(w,x,y,z) \in \mathbb{Z}^4 \mid w - x,\ x - y, \text{ and } y - z \text{ are divisible by } 4\}.$$

Find integers $d_1, d_2, d_3, d_4 > 0$ and vectors $u_1, u_2, u_3, u_4 \in \mathbb{Z}^4$ such that $\{u_1, u_2, u_3, u_4\}$ is a basis for F and $\{d_1u_1, d_2u_2, d_3u_3, d_4u_4\}$ is a basis for G. [It is possible to do this using matrix methods, but intelligent trial and error is likely to be easier.] It follows that $G \subseteq F$, so we can form the factor group F/G. Deduce a description of F/G as a direct sum of cyclic groups.

11. Row and column operations

Q40: Reduce the following matrix over $\mathbb{C}[x]$ to column echelon form.

$$A = \begin{pmatrix} x-1 & x & x+1 \\ x & 0 & x \\ x+1 & x & x-1 \end{pmatrix}.$$

Q41: Consider the following matrix over $\mathbb{C}[x]$:

$$A = \begin{pmatrix} x & 0 & -1 \\ -1 & x & -1 \\ 0 & -1 & x+1 \end{pmatrix}.$$

Let M be the quotient of $\mathbb{C}[x]^3$ by the span of the columns of A.

(a) Reduce A to normal form by row and column operations.
(b) Give a polynomial $f(x)$ such that $M \simeq \mathbb{C}[x]/f(x)$.
(c) Give a list of basic $\mathbb{C}[x]$-modules whose direct sum is isomorphic to M.

Q42: Consider the following matrix over $\mathbb{C}[x]$:

$$A = \begin{pmatrix} x & 0 & 1 \\ 1 & x & 0 \\ 0 & 1 & x \end{pmatrix}.$$

Let M be the quotient of $\mathbb{C}[x]^3$ by the span of the columns of A.

(a) Reduce A to normal form by row and column operations.
(b) Give a polynomial $f(x)$ such that $M \simeq \mathbb{C}[x]/f(x)$.

Q43: Consider the following matrix over $\mathbb{C}[x]$:

$$A = \begin{pmatrix} x & 0 & 0 \\ 0 & x-1 & 0 \\ 0 & 0 & x^2-x \end{pmatrix}.$$

Let M be the quotient of $\mathbb{C}[x]^3$ by the span of the columns of A.

(a) Reduce A to normal form by row and column operations.
(b) Describe M as a direct sum of cyclic modules over $\mathbb{C}[x]$.

Q44: Let M be the subgroup of $\mathbb{Z}^5$ generated by the columns of the following matrix.

$$A = \begin{pmatrix} 1 & 1 & 2 & 3 & 3 \\ 1 & -1 & 0 & 1 & -1 \\ 1 & 1 & 2 & -1 & -1 \\ 1 & 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & -1 & 1 \end{pmatrix}$$

Determine the structure of $\mathbb{Z}^5/M$.

[Those of you studying representation theory may be interested to know that A is the character table of the symmetric group S_4. This means that $\mathbb{Z}^5/M = C/R$, where R is the character ring of S_4 and C is the ring of integer-valued class functions. You can do the question without knowing any of this, however.]

Q45: Reduce the following matrix over $\mathbb{Z}$ to column echelon form.

$$A = \begin{pmatrix} 71 & 97 & 113 & 149 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

(The entries in the top row are the 20'th, 25'th, 30'th and 35'th prime numbers. You could actually derive the answer from this fact, but it is probably easier to just do the calculation.)

Q46:

(a) Let A be and $n \times n$ matrix over $\mathbb{Z}$, and suppose that B is obtained from A by row and column operations. Show that $|\det(A)| = |\det(B)|$.
(b) Let A be an $n \times n$ matrix over $\mathbb{Z}$, and suppose that $|\det(A)| = d \neq 0$. Let M be the quotient of $\mathbb{Z}^n$ by the span of the columns of A. Show that M is a finite Abelian group of order d.
(c) Let A be an $n \times n$ matrix over $\mathbb{C}[x]$, with $\det(A) = f(x) \neq 0$. Let M be the quotient of $\mathbb{C}[x]^n$ by the span of the columns of A. Show that $\dim_{\mathbb{C}}(M)$ is the degree of the polynomial $f(x)$.

Q47: Let A be the 3×3 matrix whose (i,j)'th entry is $1/(i+j-1)$, as shown below:

$$A = \begin{pmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{pmatrix}.$$

This is called a *Hilbert matrix.* Put $B = 120A$ (so the entries in B are integers) and let M be the quotient of $\mathbb{Z}^3$ by the span of the columns of B. Describe M as a direct sum of cyclic groups.

Q48: Let M be the subgroup of $\mathbb{Z}^2$ generated by the vectors (k, k^3) for $k = 2, 3, 4, 5$. Find a basis for M over $\mathbb{Z}$, and determine the order of $\mathbb{Z}^2/M$.

Q49: Let A be the following matrix over $\mathbb{Z}$:

$$A = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 15 \end{pmatrix}.$$

(a) Reduce A to normal form by row and column operations over $\mathbb{Z}$.
(b) What does the answer tell you about the group $\mathbb{Z}_6 \oplus \mathbb{Z}_{10} \oplus \mathbb{Z}_{15}$?
(c) Can you prove this fact in an easier way?

12. Primary decomposition

Q50: Write the $\mathbb{C}[x]$-module $M := \mathbb{C}[x]/(x^4 - x^2) \oplus \mathbb{C}[x]/(x^4 - 2x^2 + 1)$ as a direct sum of basic $\mathbb{C}[x]$-modules.

Q51: Let M be the $\mathbb{Z}$-module $\mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_9$. List all the elements of the subgroups $F_2^1(M)$, $F_2^2(M)$ and $F_3^1(M)$.

Q52:

(a) List all the Abelian groups of order 225 up to isomorphism. You should write all the groups as direct sums of basic $\mathbb{Z}$-modules.
(b) Which of the groups in your list is isomorphic to $\mathbb{Z}_{225}$?
(c) Let M be an Abelian group of order 225. Suppose that there is an element in M of order 25, and that there are 9 elements $x \in M$ satisfying $3x = 0$. Which of the groups in your list is isomorphic to M?

Q53: Let p be a prime number.

(a) List all the Abelian groups of order p^4, up to isomorphism.

(b) Let M be an Abelian group of order p^4, and put $N = \{x \in M \mid px = 0\}$. Suppose that $\dim_{\mathbb{Z}/p} N = 3$. Which of the groups in your list is isomorphic to M?
(c) How many isomorphism classes of Abelian groups of order 10000 are there? You should justify your answer, but you need not list all the groups.

Q54:

(a) Given a prime number p, list all the isomorphism classes of Abelian groups of order p^5.
(b) Let M be an Abelian group of order 2^5. Suppose that M has precisely 8 elements satisfying $2m = 0$, but that all elements satisfy $4m = 0$. Which of the groups in your list is isomorphic to M?

Q55: Suppose that $f \in C^\infty(\mathbb{R}, \mathbb{R})$ satisfies $f''' = f'$. Find elements $e_0, e_1, e_{-1} \in \mathbb{R}[D]$ such that

$$\begin{aligned} f &= e_{-1}f + e_0 f + e_1 f \\ (D-1)e_{-1}f &= 0 \\ De_0 f &= 0 \\ (D+1)e_1 f &= 0. \end{aligned}$$

Q56: In this problem we solve the differential equation $f''' = f'$.

(a) Write the equation $f''' = f'$ in the form $p(D)f = 0$ for some polynomial p.
(b) Put $M = \{f \in C^\infty(\mathbb{R}, \mathbb{R}) \mid f''' = f'\}$. Factorise $p(D)$ and thus write M as a direct sum of three submodules.
(c) Show that if $f''' = f'$ then there are constants $a, b, c \in \mathbb{R}$ such that $f(t) = ae^t + be^{-t} + c$ for all t.

Q57: Consider $C^\infty(\mathbb{R}, \mathbb{C})$ as a module over $\mathbb{C}[D]$. Let W be the space of functions of the form $p_1(t)e^{\lambda_1 t} + \ldots + p_r(t)e^{\lambda_r t}$ for some $\lambda_1, \ldots, \lambda_r \in \mathbb{C}$ and polynomials $p_1, \ldots, p_r \in \mathbb{C}[t]$. In this (quite substantial) problem we will show that $\mathrm{tors}(C^\infty(\mathbb{R}, \mathbb{C})) = W$.

(a) For each $\lambda \in \mathbb{C}$, let W_λ be the space of functions of the form $f(t) = p(t)e^{\lambda t}$, where p is polynomial. Calculate $(D - \lambda)^k f$, and deduce that W_λ is a torsion module over $\mathbb{C}[D]$.
(b) Suppose that $f \in C^\infty(\mathbb{R}, \mathbb{C})$ and that $(D-\lambda)^k f = 0$ for some $k \geq 0$. Prove that the function $g(t) := f(t)e^{-\lambda t}$ satisfies $D^k g = 0$.
(c) Suppose that a function $g \in C^\infty(\mathbb{R}, \mathbb{C})$ satisfies $D^k g = 0$. Prove by induction on k that g is a polynomial of degree less than k.
(d) Deduce that every function f with $(D - \lambda)^k f = 0$ lies in W_λ.
(e) Suppose that $f \in C^\infty(\mathbb{R}, \mathbb{C})$ and that $p(D)f = 0$ for some nonzero element $p(D) \in \mathbb{C}[D]$. By factoring $p(D)$ and considering the module $\{g \in C^\infty(\mathbb{R}, \mathbb{C}) \mid p(D)g = 0\}$, show that $f \in W_{\lambda_1} + \ldots + W_{\lambda_r}$ for some r.
(f) Deduce that $\mathrm{tors}(C^\infty(\mathbb{R}, \mathbb{C})) = W$.

Q58: Consider the following matrix over $\mathbb{C}[x]$:

$$A = \begin{pmatrix} x^3 & x^2 & x \\ x & x^2 & x \\ x & x & x \end{pmatrix}.$$

Let M be the quotient of $\mathbb{C}[x]^3$ by the span of the columns of A.

(a) Reduce A to normal form by row and column operations.
(b) Give a list of three cyclic $\mathbb{C}[x]$-modules whose direct sum is isomorphic to M.
(c) Gve a list of basic $\mathbb{C}[x]$-modules whose direct sum is isomorphic to M.

13. Canonical forms for square matrices

Q59: Let A be the matrix

$$\begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}.$$

We assume that $a, b, c \in \mathbb{R}$, but we consider A as a matrix over $\mathbb{C}$ so we get a module M_A over $\mathbb{C}[x]$. Prove that

$$M_A \simeq M_0 \oplus M_{ir} \oplus M_{-ir},$$

where $r = \sqrt{a^2 + b^2 + c^2}$.

Q60: (This question is quite elaborate.) A 3×3 circulant matrix is a matrix of the following form:

$$A = \begin{pmatrix} u & v & w \\ v & w & u \\ w & u & v \end{pmatrix}.$$

We will assume that u, v and w are real numbers. Put $a = u + v + v$ and $b = ((u - v)^2 + (v - w)^2 + (w - u)^2)/2$.

(a) Show that the characteristic polynomial of A is $(x - a)(x^2 - b)$ [You may wish to start by performing some row and column operations rather than just wading in and calculating the determinant.]
(b) Show that if $uv + vw + wu \neq 0$ and u, v, w are not all the same then the minimal polynomial is equal to the characteristic polynomial.
(c) Show that if $uv + vw + wu = 0$ and $a \neq 0$ then the minimal polynomial is $x^2 - a^2$.
(d) Calculate and factorise the minimal polynomial when $u = -13$, $v = 11$ and $w = 10$.
(e) Calculate and factorise the minimal polynomial when $u = -2$, $v = 3$ and $w = 6$.

Q61: Consider the following matrix over $\mathbb{C}$:

$$A = \begin{pmatrix} -1 & 1 & 1 & -1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

(a) What is the characteristic polynomial of A?
(b) What is the minimal polynomial of A?
(c) What is the rank of the matrix $A + I$?
(d) Which direct sum of basic $\mathbb{C}[x]$-modules is isomorphic to M_A?

Q62: Consider the following matrix over $\mathbb{C}$:

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(a) What is the characteristic polynomial of A?
(b) What is the minimal polynomial of A?
(c) Which direct sum of basic $\mathbb{C}[x]$-modules is isomorphic to M_A?
(d) What is the Jordan normal form of A?

Q63: Consider the following matrix over $\mathbb{C}$:

$$A = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}.$$

(a) What is the characteristic polynomial of A?
(b) What are the ranks of $A+I$ and $A-I$?
(c) What is the Jordan normal form of A?
(d) Show that M_A is cyclic.

Q64: Put $\alpha = \sqrt{-8}$. Consider the following matrix over $\mathbb{C}$:

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & \alpha \\ 0 & 1 & \alpha & -4 \\ 1 & \alpha & -4 & -\alpha \end{pmatrix}.$$

(a) What is the characteristic polynomial of A?
(b) What are the ranks of $A+I$ and $A-I$?
(c) What is the Jordan normal form of A?
(d) Show that M_A is cyclic.