# Categories and functors

In various courses you will have studied different kinds of mathematical structures: groups, rings, modules, vector spaces (with or without inner products), metric spaces, surfaces and so on. Generically we refer to these structures as *objects*. You will have noticed various parallels between these structures:

- In each case we have a notion of structure-preserving maps: homomorphisms between groups, rings or modules, linear maps between vector spaces, continuous maps between metric spaces, diffeomorphisms between surfaces, and so on. Generically we refer to structure-preserving maps as *morphisms*.
- In all cases, we can consider the image of a morphism as a new object (the image of a group homomorphism is a group, for example). In some of the cases we can also consider the kernel of a morphism as a new object, but this does not make sense for metric spaces or surfaces.
- In most cases, we can take the product of two objects $X$ and $Y$ to get a new object called $X \times Y$. This does not work for surfaces, because the product of two surfaces is 4-dimensional and so is not itself a surface.

The main purpose of category theory is to organise and study these similarities and differences. A *category* is a system of objects and morphisms satisfying various axioms. For example, we have a category of finite groups, a category of commutative rings, a category of compact metric spaces, and so on. Categories are a basic part of the language of much research in pure mathematics, and are also important in logic and theoretical computer science.

One of the central concepts is that of a *functor*, which is a construction that converts objects in one category to objects in another category. For example, given a group $G$ we let $G'$ denote the subgroup generated by all elements of the form $ghg^{-1}h^{-1}$ and put $G_{ab} = G/G'$. This is an abelian group, called the *abelianisation* of $G$. Abelianisation is a functor from the category of all groups to the category of abelian groups. Similarly, in algebraic topology, for any based space $X$ we have a fundamental group $\pi_1(X)$. It turns out that $\pi_1$ defines a functor from based spaces to groups.

The project should start by defining categories, functors and natural transformations, and showing how various topics from earlier courses fit into this framework. There are various ways one could continue from that point, which the supervisor will be happy to discuss.

## References


[1] F. William Lawvere and Stephen H. Schanuel, *Conceptual mathematics*, Cambridge University Press, Cambridge, 1997. A first introduction to categories; Reprint of the 1991 original.MR1485776 (99e:18001)

[2] Saunders Mac Lane, *Categories for the working mathematician*, 2nd ed., Graduate Texts in Mathematics, vol. 5, Springer-Verlag, New York, 1998.MR1712872 (2001j:18001)

[3] Benjamin C. Pierce, *Basic category theory for computer scientists*, Foundations of Computing Series, MIT Press, Cambridge, MA, 1991.MR1120026 (93c:18002)



**Supervisor: Professor Neil Strickland**
**Room J26, N.P.Strickland@shef.ac.uk**

October 14, 2008