De Rham cohomology

Given a suitable space X (for example, an open subset of $\mathbb{R}^n$) one can define vector spaces $H^k(X)$ for integers $k \geq 0$, called the *de Rham cohomology groups* of X. If X is an open subset of $\mathbb{R}^3$, then the definition can be phrased in terms of the usual operations of vector calculus (**div**, **grad** and **curl**), giving a close relationship with some physical phenomena in electromagnetism. The same circle of ideas also gives a nice unified picture of results such as the Fundamental Theorem of Calculus, Green's Theorem, Stokes' Theorem, and the Divergence Theorem. The dimensions of the spaces $H^k(X)$ give information about the topological structure of X, such as the number of holes. With more work, one can generalise everything to open subsets of $\mathbb{R}^n$ for $n > 3$. The main problem is to generalise the definitions of the dot and cross products, a topic known as *exterior algebra*. One also wants to define de Rham cohomology for sets such as curves and surfaces (which are closed, rather than open, in $\mathbb{R}^3$) and higher-dimensional generalisations (known as *submanifolds* of $\mathbb{R}^n$).

Another important topic is the multiplicative structure. Given $a \in H^i(X)$ and $b \in H^j(X)$ there is a natural definition of a product $ab \in H^{i+j}(X)$. This makes the vector space $H^*(X) = \bigoplus_i H^i(X)$ into a ring. As well as setting up the general theory, one can give a complete description of $H^*(X)$ for many interesting spaces (such as spheres, the torus, configuration spaces and so on). There are many applications, including the Brouwer Fixed Point Theorem: any continuous map $f\colon B^n \to B^n$ has a fixed point (where $B^n = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\| \leq 1\}$).

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October 14, 2008