The Lorentz Group

The (proper, orthochronous) Lorentz group L is the group of 4×4 invertible matrices A over the real numbers with the properties that $A_{11} > 0$, $\det(A) > 0$ and $A^T JA = J$, where

$$J = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

This group is of fundamental importance in the theory of relativity; it is the analogue for four-dimensional space-time of the group SO_3 of rotations of three-dimensional space. The aim of the project is to understand the algebraic, geometric, topological and physical properties of L. (The amount of physics involved is up to the student; it could be minimal, or it could be the central theme.) A fact that should certainly be included is that L is isomorphic to the group $M = PSL_2(\mathbb{C})$ of Möbius transformations, and that the action of M on the Riemann sphere is essentially the same as the action of L on the "celestial sphere". This can be used to explain Penrose's beautiful observation that a relativistically moving sphere continues to appear spherical despite relativistic distortions. A topologically inclined student could also deduce that L is homeomorphic to the space

$$F_3S^2 = \{(u, v, w) \mid u, v \text{ and } w \text{ are three distinct points on } S^2\} \subset S^2 \times S^2 \times S^2,$$

and also that L is homotopy equivalent to $\mathbb{R}P^3$. A student with more algebraic interests could classify the elements of L up to conjugacy, or explain some of the representation theory of L.

The total amount of material in the project as proposed is in fact relatively small; however, there does not seem to be a single good source for it. The student will need to collect small fragments from a number of different sources, some of which are listed below.

References

[1] R. Carter, G. Segal, and I. Macdonald. *Lectures on Lie groups and Lie algebras.* Cambridge University Press, Cambridge, 1995. With a foreword by Martin Taylor.

[2] S. A. Huggett and K. P. Tod. *An introduction to twistor theory.* Cambridge University Press, Cambridge, second edition, 1994.

[3] T. Needham. *Visual complex analysis.* The Clarendon Press Oxford University Press, New York, 1997.

[4] R. Penrose and W. Rindler. *Spinors and space-time. Vol. 1.* Cambridge University Press, Cambridge, 1984. Two-spinor calculus and relativistic fields.

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