Soluble groups

A *subnormal series* for a finite group G is a chain of subgroups

$$\{1\} = H_0 \leq H_1 \leq \cdots \leq H_{r-1} \leq H_r = G$$

such that H_{i-1} is a normal subgroup of H_i (for $i = 1, \ldots, r$); this means that we can define quotient groups H_i/H_{i-1}. We say that G is *soluble* if there exists a subnormal series for which all the quotient groups H_i/H_{i-1} are abelian.

Soluble groups are easier to handle than more general groups, because many things can be reduced to the abelian case. Solubility is important in Galois theory, because the roots of a polynomial can be expressed in terms of radicals iff the corresponding Galois group is soluble. A very important theorem, far beyond the scope of this project, says that all groups of odd order are soluble; the proof takes hundreds of pages.

This project should start with some generalities about subnormal series. In particular, a *composition series* for a group G is a subnormal series all of whose quotients are simple groups; these quotients are called the *composition factors* of the group. The Jordan-Hölder theorem says that they are well-defined, despite the fact that G will typically have many different composition series; this should be proved in the project.

Cyclic and dihedral groups are soluble. The symmetric and alternating groups S_n and A_n are soluble for $n \leq 4$, but not for $n \geq 5$. Subgroups and quotient groups of soluble groups are soluble. A group of order p^a (with p prime) is always soluble, by a fairly easy argument. A group of order $p^a q^b$ (with p and q both prime) is also soluble, by a theorem of Burnside, which should be accessible to a student who has studied Representation Theory.

The following theorem of Hall could form the main goal of the project:

Theorem: *Suppose that G is soluble and $|G| = mn$ with m and n coprime; then G has a subgroup of order m, and any two such groups are conjugate.*

(The case where m is a power of a prime is a theorem of Sylow, which holds even when G is not soluble. When m is not a power of a prime, solubility is definitely needed.)

References

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