SOME ALGEBRAIC EXAMPLES IN TOPOLOGY

NEIL STRICKLAND

A natural starting point for the examples presented here is the following question: what is the right topology to use when considering problems involving only polynomial functions ? For simplicity, let's restrict ourselves to the complex case. To answer the question, we need to know some properties of the ring $A_n = \mathbb{C}[z_1, \dots z_n]$ of all polynomial functions from $\mathbb{C}^n$ to $\mathbb{C}$. Typical elements of this ring are $(2+3i)z_2^5 z_3 + z_1^2$ and so on.

A fundamental fact is the following special case of Hilbert's Basis Theorem:

Theorem 1. *The ring A_n is Noetherian; that is, every ideal is finitely generated.*

We shall need to know a number of other things about the ideals in A_n. Firstly, consider a point $a = (a_1, \dots a_n) \in \mathbb{C}^n$. There is a ring homomorphism $\epsilon_a \colon A_n \to \mathbb{C}$ defined by $\epsilon_a(f) = f(a)$. Of course, it is a more usual situation that we have some fixed function f and we think of $f(a)$ as the value of that function at a variable point a. Here, however, we have fixed the point a and we are allowing f to vary; we regard $f(a)$ as the value of the function $\epsilon_a \colon A_n \to \mathbb{C}$ at the point $f \in A_n$. It is an absolute triviality that this is a ring homomorphism, once one understands what that means. We also have a ring homomorphism $\eta \colon \mathbb{C} \to A_n$ sending $z \in \mathbb{C}$ to the constant function whose only value is z; in other words, $\eta(z)(a) = z$ for all a.

Note that $\epsilon_a \circ \eta$ is the identity map $1_{\mathbb{C}}$, so ϵ_a is a surjective map from A_n to a field $\mathbb{C}$. It follows that the kernel $\mathfrak{m}_a = \{f \in A_n \mid f(a) = 0\}$ is a maximal ideal.

Another important fact is the Weak Nullstellensatz:

Theorem 2. *Every maximal ideal $\mathfrak{m} < A_n$ has $\mathfrak{m} = \mathfrak{m}_a$ for a unique point $a \in \mathbb{C}^n$. In other words, there is a bijection between $\mathbb{C}^n$ and the set $\max(A_n)$ of maximal ideals in A_n.*

Next, we need to consider sets defined by polynomial equations. We shall convert all equations $g(a) = h(a)$ into the form $f(a) = g(a) - h(a) = 0$. The idea is that we want to start with some polynomials $f_1, \dots f_m \in A_n$ and consider the set

$$X = V(f_1, \dots f_m) = \{a \mid f_1(a) = f_2(a) = \dots f_m(a) = 0\}$$

On the other hand, we can consider an arbitary set $X \subset \mathbb{C}^n$ and try to define it by polynomial equations in this way. Most sets X cannot be so defined, of course; for example, the only such subsets of $\mathbb{C}$ are finite or the whole space. We may as well consider all possible equations which are true on X, and then ask whether there are any unwanted points a not in X which nonetheless satisfy all the equations. This leads us to consider

$$I(X) = \{f \in A_n \mid a \in X \Rightarrow f(a) = 0\} \leq A_n$$

For example if $X = \mathbb{Z} \times \{0\} \subset \mathbb{C}^2$, then the only functions $f(z_1, z_2)$ with $f(n, 0) = 0$ for all $n \in \mathbb{Z}$ are those of the form $f(z_1, z_2) = z_2 g(z_1, z_2)$, so they also vanish at $(z, 0)$ even when z is not an integer. Thus, X is not polynomially definable.

Note that $I(X)$ is actually an ideal in A_n. It can also be described as

$$I(X) = \bigcap_{a \in X} \mathfrak{m}_a$$

This leads us to remark that the set $Y = V(f_1, \dots f_n)$ really only depends on the ideal generated by $\{f_1, \dots f_n\}$. If f lies in this ideal then $f = 0$ on Y also. For any ideal J we can consider

$$V(J) = \{a \in \mathbb{C}^n \mid f \in J \Rightarrow f(a) = 0\}$$

For any finite list of polynomials $\{f_1, \dots f_m\}$, we can consider the generated ideal $J = (f_1, \dots f_m)$ and then $V(J) = V(f_1, \dots f_m)$. On the other hand, given an ideal J we can (by Hilbert's basis theorem) choose elements $f_1, \dots f_m$ which generate it, and then $V(f_1, \dots f_m) = V(J)$. From now on, we shall use only the newer notation.

We can now define the Zariski topology on $\mathbb{C}^n$: the closed sets are precisely the sets $V(J)$ for the various different ideals $J \leq A_n$. To see that this indeed defines a topology, we need the following facts:

$$\begin{aligned} \emptyset &= V(A_n) \\ \mathbb{C}^n &= V(0) \\ V(I) \cup V(J) &= V(I \cap J) \\ \bigcap_k V(J_k) &= V(\sum_k J_k) \end{aligned}$$

All of these are easy to prove; the third perhaps requires some comment. Suppose that $a \in V(I) \cup V(J)$. Now suppose that $f \in I \cap J$. Either $a \in V(I)$, so $f(a) = 0$ because $f \in I$; or $a \in V(J)$ and $f(a)$=0 because $f \in J$. Thus $a \in V(I \cap J)$. On the other hand, suppose $a \notin V(I) \cup V(J)$. Because $a \notin V(I)$, there is some $f \in I$ with $f(a) \neq 0$. Similarly, there is some $g \in J$ with $g(a) \neq 0$. Thus $fg \in I \cap J$ and $fg(a) \neq 0$. Thus $a \notin V(I \cap J)$. This shows that $V(I) \cup V(J) = V(I \cap J)$ as claimed.

Here are some facts you can prove. I refer to the Zariski topology throughout, of course, unless I specifically state otherwise.

(1) Every one-point set $\{a\}$ is closed.
(2) The sets $D(f) = \{a \mid f(a) \neq 0\}$ form a basis for the Zariski topology.
(3) $D(f) \cap D(g) = D(fg)$.
(4) $\mathbb{C}^n$ is not Hausdorff.
(5) If we give $\mathbb{C}$ the Zariski topology, then the resulting product topology on $\mathbb{C}^2$ is different from the Zariski topology. (Consider the diagonal, for example).
(6) A function $f\colon \mathbb{C}^n \to \mathbb{C}^m$ given by polynomial functions is continuous. Careful; remember what you have just proved.
(7) Every nonempty open set is dense.
(8) Every nonempty open set is connected.

Here is an interesting application of the above. Consider a 2×2 complex matrix

$$C = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

The eigenvalues of C are the roots of the characteristic polynomial

$$p_C(\lambda) = \det(C - \lambda I) = \lambda^2 - \tau\lambda + \delta$$

$$\tau = a + d \qquad \delta = ad - bc$$

We find that there is a repeated root iff the discriminant $\Delta(a, b, c, d)$ vanishes, where

$$\Delta = \tau^2 - 4\delta = (a + d)^2 - 4(ad - bc)$$

Thus, the set $V(\Delta)$ of matrices with a repeated eigenvalue is a Zariski closed subset of $M_2(\mathbb{C}) \simeq \mathbb{C}^4$. The complement $D(\Delta)$ is open and nonempty, therefore dense. The same is true for an arbitary value of 2, although some work needs to be done to define the discriminant.

Now consider the function $f\colon M_2(\mathbb{C}) \to M_2(\mathbb{C})$ defined by

$$f(C) = p_C(C) = C^2 - \tau C + \delta I$$

The Cayley-Hamilton theorem asserts that $f(C) = 0$ for any C. This is easy if C has distinct eigenvalues, as we can then diagonalise everything. On the other hand, f is Zariski continuous, so $f^{-1}\{0\}$ is closed and contains the dense set $D(\Delta)$, so $f^{-1}\{0\} = M_2(\mathbb{C})$ and $f = 0$ as required.

To return to the general theory: note that $I(X)$ is not an entirely arbitary ideal. It has the following extra property (why ?):

$$f^m \in I(X) \Rightarrow f \in I(X)$$

this leads us to the following definition:

$$\sqrt{J} = \text{ the radical of } J = \{f \in A_n \mid f^m \in J \text{ for some } m \in \mathbb{N}\}$$

this is again an ideal. The main point in seeing this is that when expanded out, $(f+g)^{m+l}$ involves only terms divisible either by f^m or by g^l. Note also that $\sqrt{\sqrt{J}} = \sqrt{J}$.

An ideal J is said to be a radical ideal iff $J = \sqrt{J}$ iff $J = \sqrt{K}$ for some ideal K.

The full story on the correspondence between the topology and the algebra is as follows:

$$I(V(J)) = \sqrt{J}$$
$$V(I(X)) = \overline{X}$$

and the maps $X \mapsto I(X)$ and $J \mapsto V(J)$ give a bijection between closed sets and radical ideals. The equation $I(V(J)) = \sqrt{J}$ is essentially Hilbert's Nullstellensatz. A "nullstelle" is a point where a function is zero and a "satz" is a theorem. Why this has kept its German name is a mystery. There is another theorem of Hilbert in Galois theory which is for some equally mysterious reason always referred to as "Hilbert's Theorem 90". Curious.

Using the above, you can show that

(1) $\mathbb{C}^n$ is compact.
(2) $D(f)$ is compact.
(3) Every open subset is compact.