TOPOLOGY PROBLEM ANSWERS

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(1)

(2) (a) The sets

$$U(n,\underline{b}) = \{b_1\} \times \{b_2\} \times \dots \{b_n\} \times \prod_{m>n} \mathbf{2}$$

are elements of the standard basis for the product topology, and so are certainly open. The complement of $U(n,\underline{b})$ is the union of the sets $U(n,\underline{c})$ for all the elements $\underline{c} \in \mathbf{2}^n$ for which $\underline{c} \neq \underline{b}$. This complement is thus open, so $U(n,\underline{b})$ is also closed.
Consider two sets $V = U(n,\underline{b})$ and $W = U(m,\underline{c})$, say. Suppose wlog that $n \leq m$. If $\underline{c}[n] = \underline{b}$ then $W \subseteq V$ so $V \cap W = W$; otherwise $V \cap W = \emptyset$.
Suppose that V is one of the sets in the standard subbasis for the product topology, so $V = \pi_n^{-1}(b)$ say, for some $b \in \mathbf{2}$. Then V is the union of those sets $U(n,\underline{b})$ for which $b_n = b$:

$$V = \bigcup \{U(n,\underline{b}) \mid b_n = b\}$$

This is enough to show that the sets $U(n,\underline{b})$ form a basis.

(b) If this were false, then we would have

$$U(1,0) \subseteq \bigcup_{i \in J_0} U_i$$

$$U(1,1) \subseteq \bigcup_{i \in J_1} U_i$$

for some finite subsets J_0 and J_1 of I. Then $J = J_0 \cup J_1$ would be finite and we would have

$$X = U(0,0) \cup U(1,1) \subseteq \bigcup_{i \in J} U_i$$

contrary to the hypothesis that X is not covered by any finite subfamily of $\mathcal{U}$. For the rest of this question I shall abbreviate this to "is not finitely covered".

(c) Suppose we have chosen $(a_1, \dots a_n) \in \mathbf{2}^n$ such that $U(n,(a_1,\dots a_n))$ is not finitely covered. By the same logic as above, one of the two sets $V_0 = U(n,(a_1,\dots a_n,0))$ and $V_1 = U(n,(a_1,\dots a_n,1))$ is not finitely covered. If V_0 is finitely covered, then we take $a_{n+1} = 1$, and otherwise we take $a_{n+1} = 0$. Continuing in this way, we choose a_n for all n. Note that we have a definite rule for choosing a_{n+1}, so we are not using the axiom of choice here (not that it would worry me if we were).

(d) Suppose we have chosen $\underline{a}$ as above. As $\mathcal{U}$ is a covering of X, we have $\underline{a} \in U_i$ for some i. As U_i is open and the sets $U(n,\underline{b})$ form a basis, we have

$$\underline{a} \in U(n,\underline{b}) \subseteq U_i$$

for some n and $\underline{b}$. As $\underline{a} \in U(n,\underline{b})$, we must have $\underline{b} = \underline{a}[n]$. Thus $U(n,\underline{a}[n])$ is contained in the single set U_i, hence certainly finitely covered, contrary to the construction.

(3) Suppose that $X \subset \mathbb{R}^n$ is such that $\overline{X}$ is disconnected. There are then subsets U and V of $\mathbb{R}^n$ such that:
(a) $\overline{X} \subset U \cup V$
(b) $\overline{X} \cap U \cap V = \emptyset$
(c) $\overline{X} \cap U \neq \emptyset \neq \overline{X} \cap V$

The claim is that the corresponding statements hold when $\overline{X}$ is replaced by X. In the first two cases, this follows trivially from the fact that $X \subseteq \overline{X}$. For the third case, choose $x \in \overline{X} \cap U$. Then $x \in U$, so U is a neighbourhood of x. Also, $x \in \overline{X}$, which means that every neighbourhood of x meets X, so that $U \cap X \neq \emptyset$ as required. Similarly, $V \cap X \neq \emptyset$. All this implies that X is also disconnected.

We have proved that $\overline{X}$ disconnected implies X disconnected. Equivalently, X connected implies $\overline{X}$ connected.

(4) Write q for the quotient map $X \to Y$. First, $Y = q(X)$ is a continuous image of X and therefore compact. Next, suppose y and y' are distinct points of Y. Then $y = q(x)$ and $y' = q(x')$ say, where $x \not\sim x'$. This means that there is a function $f \in A$ with $f(x) \neq f(x')$, say $a = f(x) < a' = f(x')$. Clearly (by the definition of $\sim$)

$$z \sim z' \Rightarrow f(z) = f(z')$$

so f induces a continuous map $\tilde{f}\colon Y \to \mathbb{R}$ with $\tilde{f} \circ q = f$. In particular,

$$\tilde{f}(y) = \tilde{f}(q(x)) = f(x) = a < a' = f(x') = \tilde{f}(q(x')) = \tilde{f}(y)$$

Thus

$$U = \tilde{f}^{-1}((-\infty, (a + a')/2))$$

and

$$U' = \tilde{f}^{-1}((a + a')/2, \infty))$$

are disjoint neighbourhoods of y and y'. Thus Y is Hausdorff.

Consider

$$\tilde{A} = \{\tilde{f} \mid f \in A\} \subseteq C(Y)$$

This is a subalgebra, and it separates Y by the definition of $\sim$. Because q is surjective, we have

$$\|\tilde{f}\| = \sup_y |\tilde{f}(y)| = \sup_x |\tilde{f}(q(x))| = \sup_x |f(x)| = \|f\|$$

This shows that the map $f \mapsto \tilde{f}$ is an isometric isomorphism $A \to \tilde{A}$. Now, A is closed in the complete space $C(X)$ so A is complete. Moreover, $\tilde{A}$ is isometrically isomorphic to A and hence complete, and hence closed in $C(Y)$. Thus, by Stone-Weierstrass, $\tilde{A} = C(Y)$. We conclude that $C(Y)$ is isometrically isomorphic to A.

(5) Define a (discontinuous) function

$$S\colon C(\mathbb{N}) \to \mathcal{P}(\mathbb{N}) = \{ \text{ subsets of } \mathbb{N}\}$$
$$S(f) = \{n \mid f(n) < 1/2\}$$

Recall that $\mathcal{P}(\mathbb{N})$ is uncountable. Suppose that $A \subset C(\mathbb{N})$ is countable, so $S(A)$ is also countable and thus not the whole of $\mathcal{P}(\mathbb{N})$. Choose a set $T \subseteq \mathbb{N}$ such that $T \neq S(f)$ for any $f \in A$, and write

$$g(n) = \begin{cases} 0 & \text{if } n \in T \\ 1 & \text{otherwise} \end{cases}$$

I claim that $d(f, g) \geq 1/2$ for any $f \in A$, so that $g \notin \overline{A}$ and thus A is not dense. Indeed, if $f \in A$ then $S(f) \neq T$ so either
(a) $\exists n \in S(f) \setminus T$ so $f(n) < 1/2$ and $g(n) = 1$.
or
(b) $\exists n \in T \setminus S(f)$ so $f(n) \geq 1/2$ and $g(n) = 0$.

This immediately implies the claim.

(6) Write

$$\sigma' = \{S(K, U) \mid K \subseteq X \text{ compact }, U \text{ open in } Y\}$$

Let τ and τ' be the topologies generated by σ and σ', so τ' is by definition the compact-open topology. We are required to show that $\tau = \tau'$. Clearly $\sigma \subseteq \sigma'$ and therefore $\tau \subseteq \tau'$. We need only show that $\sigma' \subseteq \tau$; for then τ', being the smallest topology containing σ', will be contained in τ.

Consider $f \in S(K,U) \in \sigma'$, so $f(K) \subseteq U$ and $K \subseteq f^{-1}(U)$. As $U \subseteq Y$ is open, it can be written as $U = \bigcup_i U_i$ with $U_i \in \beta$. The open sets $f^{-1}(U_i)$ cover the compact set K, so

$$K \subseteq f^{-1}(U_{i_1}) \cup \dots f^{-1}(U_{i_n})$$

say. By the shrinking lemma we can cover K by compact sets $K_1, \dots, K_n$ with $K_k \subseteq f^{-1}(U_{i_k})$. Write

$$V = \bigcap_k S(K_k, U_{i_k}) \in \tau$$

so $f \in V$. If $g \in V$ then

$$g(K) = g(\bigcup_k K_k) = \bigcup_k g(K_k) \subseteq \bigcup_k U_{i_k} \subseteq U$$

so $g \in S(K,U)$. Thus $f \in V \subseteq S(K,U)$ and $V \in \tau$, which shows that $S(K,U)$ is a τ-neighbourhood of f. As f was an arbitary point of $S(K,U)$ we see that $S(K,U)$ is τ-open. Thus $\sigma' \subseteq \tau$ as required.

(7) Let X be a complete metric space and $Y \subseteq X$. Write i for the isometric embedding of Y in its canonical completion $\tilde{Y}$. The inclusion $j\colon Y \to X$ is an isometric embedding and X is complete so there is a unique isometric embedding $\tilde{j}\colon \tilde{Y} \to X$ with $\tilde{j} \circ i = j$.

As $\tilde{j}$ is continuous, $\tilde{j}^{-1}(\overline{Y}) \subseteq \tilde{Y}$ is closed and it contains $i(Y)$. However, $i(Y)$ is dense in $\tilde{Y}$ so $\tilde{j}^{-1}(\overline{Y}) = \tilde{Y}$ and so $\tilde{j}(\tilde{Y}) \subseteq \overline{Y}$.

On the other hand, $\tilde{j}(\tilde{Y})$ is isometrically isomorphic to the complete metric space $\tilde{Y}$, so it is complete. However, a complete subspace of a metric space is closed and $\tilde{j}(\tilde{Y}) \supseteq Y$ so $\tilde{j}(\tilde{Y}) = \overline{Y}$. Thus $\tilde{j}\colon \tilde{Y} \to \overline{Y}$ is an isometric isomorphism and thus a homeomorphism.

(8) Fix α with $0 < \alpha < 1$ and let X be the set of contraction mappings $f\colon [0,1] \to [0,1]$ of ratio α.

We want to prove that $X \subseteq C[0,1]$ is compact, so by Arzela-Ascoli it is enough to check that it is bounded, closed and equicontinuous. Boundedness is trivial. Moreover, X is equilipschitz (with $K = \alpha$) and hence equicontinuous.

We now need to prove that X is closed. Suppose $0 \leq x < y \leq 1$. Write $r = y - x$. The evaluation map

$$\mathrm{ev}_x\colon C[0,1] \to \mathbb{R} \qquad \mathrm{ev}_x(f) = f(x)$$

is continuous. Thus

$$\Delta_{xy} = \mathrm{ev}_x - \mathrm{ev}_y\colon C[0,1] \to \mathbb{R}$$

is continuous and the set

$$F_{xy} = \{f \in C[0,1] \mid |f(x) - f(y)| \leq \alpha|x - y|\} = \Delta_{xy}^{-1}([-\alpha r, \alpha r])$$

is closed in $C[0,1]$. Thus

$$X = \{f \mid \|f\| \leq 1\} \cap \bigcap_{x,y} F_{xy}$$

is also closed, as required.

Now consider the function

$$F\colon X \to [0,1]$$
$$F(f) = \text{ the unique fixed point of } f$$

Suppose $f, g \in X$ and $F(f) = x$ and $F(g) = y$. By the inequality in the last question,

$$d(x,z) \leq d(z, f(z))/(1-\alpha)$$

for any z. In particular (using $g(y) = y$) we have,

$$d(x,y) \leq d(y, f(y))/(1-\alpha) = d(g(y), f(y))/(1-\alpha) \leq \|g - f\|/(1-\alpha)$$

In other words,

$$d(F(f), F(g)) \leq d(f,g)/(1-\alpha)$$

This shows that F is continuous.

(9) Let X be a nonempty complete metric space, and let $f\colon X \to X$ be a contraction mapping with ratio $\alpha < 1$, so that

$$d(f(x), f(y)) \leq \alpha d(x, y)$$

Choose $a = a_0 \in X$ and write $r = d(a, f(a))$ and $a_{n+1} = f(a_n)$. I claim that the sequence (a_n) is Cauchy. Indeed, by induction we see that

$$d(a_n, a_{n+1}) = d(f(a_{n-1}), f(a_n)) \leq \alpha^n r$$

so if $m \leq n$ we have

$$\begin{aligned} d(a_m, a_n) &\leq d(a_m, a_{m+1}) + \dots d(a_{n-1}, a_n) \\ &\leq (\alpha^m + \dots \alpha^{n-1})r \\ &= (\alpha^m - \alpha^n)r/(1-\alpha) \end{aligned}$$

which easily implies the claim. Thus, as X is complete, the sequence converges to a limit b. Moreover

$$f(b) = f(\lim_n a_n) = \lim_n f(a_n) = \lim_n a_{n+1} = b$$

So b is a fixed point. Suppose that c is another fixed point. Then

$$d(b, c) = d(f(b), f(c)) \leq \alpha d(b, c)$$

As $\alpha < 1$, this implies $d(b, c) = 0$ and thus $b = c$. Thus b is the unique fixed point.

Finally,

$$d(a, b) = \lim_n d(a_0, a_n) \leq \lim_n \frac{1-\alpha^n}{1-\alpha} r = \frac{d(a, f(a))}{1-\alpha}$$

(10) As 0 lies in the interior of A, there exists $\epsilon > 0$ such that $B(0, \epsilon) \subseteq A$.

Suppose $0 \neq a \in A$ and $0 < t < 1$. The claim is that $B(ta, (1-t)\epsilon) \subseteq A$, so that $ta \in \mathrm{int}(A)$. Indeed, any point $b \in B(ta, (1-t)\epsilon)$ can be written as $b = ta + (1-t)u$, where $\|u\| < \epsilon$ and so $u \in A$. By convexity, $b \in A$ also, as claimed.

(b) The map f sending u to $u/\|u\|$ is continuous except at $u = 0$, which by hypothesis does not lie on the boundary of A. If we can show that it gives a bijection from $\mathrm{bdy}(A)$ to S^{n-1} then we will be done, as a continuous bijection from a compact to a Hausdorff space is a homeomorphism. Consider a unit vector $v \in S^{n-1}$. Write $B = \{t \geq 0 \mid tv \in A\}$. This can easily be seen to be bounded, closed, convex, and to contain $[0, \epsilon)$. If we write $b = \sup B$, then we conclude that $B = [0, b]$. Write $u = bv$, so $0 \neq u \in A$. As b is maximal, u cannot lie in the interior so it must be on the boundary. As $u/\|u\| = v$, our map f is surjective. All other points w with $w/\|w\| = v$ lie on the same half-line as u. Thus, to prove injectivity we must show that u is the unique point of intersection of this half-line with the boundary of A. This follows immediately from the previous part of the question.

(11) The Baire category theorem:
Let X be a compact Hausdorff space or a complete metric space. Suppose that for each $n \in \mathbb{N}$ the set $F_n \subseteq X$ is closed and has empty interior. Then $\bigcup_n F_n$ has empty interior.

Suppose that $f\colon [a, b] \to \mathbb{R}$ is continuous and injective. Write $X = f([a, b])$, which is a subspace of $\mathbb{R}^2$ and therefore Hausdorff. The map $f\colon [a, b] \to X$ is thus a continuous bijection from a compact to a Hausdorff space, and hence a homeomorphism. It follows that f gives a homeomorphism

$$f\colon [a, b] \setminus \{c\} \to X \setminus \{f(c)\}$$

for any $c \in [a, b]$. Observe that $[a, b] \setminus \{c\}$ is disconnected except when $c = a$ or $c = b$. It follows that $X \setminus \{x\}$ is disconnected except when $x = f(a)$ or $x = f(b)$.

Suppose that X has an interior point x, so the disc $B(x, \epsilon) \subset X$ for some $\epsilon > 0$. It follows that X has more than two (indeed infinitely many) interior points, so it has an interior point $y \neq f(a), f(b)$. However, by the given fact, this means that $X \setminus \{y\}$ is connected, which contradicts what we proved above. Thus the interior of X is empty.

Now suppose that $f\colon \mathbb{R} \to \mathbb{R}^2$ is continuous. By the above, $f([-n,n])$ has empty interior. By the Baire category theorem (noting that $\mathbb{R}$ is a complete metric space) we see that

$$f(\mathbb{R}) = \bigcup_n f([-n,n])$$

has empty interior. In particular, f is not surjective. Thus, there is no continuous bijection $f\colon \mathbb{R} \to \mathbb{R}^2$.

(12) (a) Suppose that $x \in Y$. Then $0 = d(x,x)$ is an element of the set $D = \{d(x,y) \mid y \in Y\}$ of nonnegative reals whose inf is $\overline{d}(x,Y)$. Thus $\overline{d}(x,Y) = 0$. Conversely, suppose that $\overline{d}(x,Y) = 0$. Then any $\epsilon > 0$ is not a lower bound for D (as 0 is the greatest lower bound). This means that $d(x,y) < \epsilon$ for some $y \in Y$, so $B(x,\epsilon) \cap Y \neq \emptyset$. As this holds for all $\epsilon > 0$, we find that $x \in \overline{Y} = Y$.

(b) First, for each $u, v \in X$ and $y \in Y$ we have

$$d(u,y) \leq d(u,v) + d(v,y)$$

Taking the inf over all $y \in Y$ we obtain

$$\overline{d}(u,Y) \leq d(u,v) + \overline{d}(v,Y)$$

so

$$\overline{d}(u,Y) - \overline{d}(v,Y) \leq d(u,v)$$

Similarly, starting with $d(v,y) \leq d(v,u) + d(u,y)$ we get

$$\overline{d}(v,Y) - \overline{d}(u,Y) \leq d(v,u) = d(u,v)$$

so

$$|\overline{d}(u,Y) - \overline{d}(v,Y)| \leq d(u,v)$$

This shows that the function $f(u) = \overline{d}(u,Y)$ is (Lipschitz and therefore) continuous.

(c) It is immediate that $e(a,a) = 0$ and $e(a,b) = e(b,a)$. Thus we need only show that

$$e(a,c) \leq e(a,b) + e(b,c)$$

We need to separate four cases. For brevity we write $P(a,b)$ to mean that $d(a,b) \leq \overline{d}(a,Y) + \overline{d}(b,Y)$ and $Q(a,b)$ to mean that $d(a,b) \geq \overline{d}(a,Y) + \overline{d}(b,Y)$. Note that $P(a,b)$ implies that $e(a,b) = d(a,b)$, and so on.

(i) Suppose that $P(a,b)$ and $P(b,c)$ hold. Then

$$e(a,c) \leq d(a,c) \leq d(a,b) + d(b,c) = e(a,b) + e(b,c)$$

(ii) Suppose $P(a,b)$ and $Q(b,c)$. Using

$$\overline{d}(a,Y) \leq d(a,b) + d(b,Y)$$

we get

$$\begin{aligned} e(a,c) &\leq \overline{d}(a,Y) + \overline{d}(c,Y) \\ &\leq d(a,b) + \overline{d}(b,Y) + \overline{d}(c,Y) \\ &= e(a,b) + e(b,c) \end{aligned}$$

(iii) The case when $Q(a,b)$ and $P(b,c)$ hold is similar.

(iv) Suppose $Q(a,b)$ and $Q(b,c)$. Then

$$\begin{aligned} e(a,c) &\leq \overline{d}(a,Y) + \overline{d}(c,Y) \\ &\leq \overline{d}(a,Y) + \overline{d}(b,Y) + \overline{d}(b,Y) + \overline{d}(c,Y) \\ &= e(a,b) + e(a,c) \end{aligned}$$

(d) Suppose that Y and Y' are disjoint closed subsets of X. By part (b), the function $g(x) = \overline{d}(x,Y) - \overline{d}(x,Y')$ is continuous, so the sets

$$U = \{x \mid g(x) < 0\} = g^{-1}((-\infty,0))$$

$$U' = \{x \mid g(x) > 0\} = g^{-1}((0,\infty))$$

are open. They are clearly disjoint, and using part (a) we see that $Y \subseteq U$ and $Y' \subseteq U'$.

(13) (a) $\mathbb{Q}$ is a countable dense subset of $\mathbb{R}$.

(b) Suppose that $X \subseteq \mathbb{R}$ is dense, and that we are given two continuous functions f and g from $\mathbb{R}$ to $\mathbb{R}$ which agree on X. Write $U = \{x \mid f(x) \neq g(x)\}$. Suppose $a \in U$. Write $\epsilon = |f(a)-g(a)|/3$. By continuity we can choose $\delta > 0$ such that $|f(x)-f(a)| < \epsilon$ and $|g(x) - g(a)| < \epsilon$ whenever $|x - a| < \delta$. It is then easy to see that $(a - \delta, a + \delta) \subseteq U$. This shows that U is open, and hence that $F = \{x \mid f(x) = g(x)\}$ is closed. As $X \subseteq F$ we have $\overline{X} = \mathbb{R} \subseteq F$, so $F = \mathbb{R}$ and $f = g$.
Alternatively: as $\mathbb{R}$ is Hausdorff, the diagonal Δ is closed, and $f \times g$ is continuous so $F = (f \times g)^{-1}(\Delta)$ is closed.

(c) Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is continuous and that $f(x) + f(y) = f(x + y)$ for all x and y in $\mathbb{R}$. First note that

$$f(0) = f(0 + 0) = f(0) + f(0) \Rightarrow f(0) = 0$$

$$f(x) + f(-x) = f(x - x) = f(0) = 0 \Rightarrow f(-x) = -f(x)$$

For nonnegative integers n we see by induction that

$$f(nx) = f((n - 1)x + x) = (n - 1)f(x) + f(x) = nf(x)$$

In fact this holds for $n < 0$ also, by the previous equation. Applying this with $x = 1/n$ we get

$$f(1/n) = f(1)/n$$

and thus, for all integers m

$$f(m/n) = mf(1)/n$$

This means that the continuous functions f and $g(x) = xf(1)$ agree on the dense subset $\mathbb{Q}$ of $\mathbb{R}$, so they are the same.

(14) Suppose X is a metric space and $A \subseteq C(X)$. Suppose that A is equilipschitz, so there is a constant K such that

$$|f(x) - f(y)| \leq Kd(x, y)$$

for all $x, y \in X$ and $f \in A$. The claim is that A is equicontinuous. Indeed, suppose that $x \in X$ and $\epsilon > 0$. Write $U = B(x, \epsilon/K) \in \mathcal{N}_x$. Then for $f \in A$ and $y \in U$ we have

$$|f(x) - f(y)| \leq Kd(x, y) < K\epsilon/K = \epsilon$$

as required.

(15) Let X be a space, A a subset of $C(X)$. Write

$$U = \bigcup\{\text{open } V \subseteq X \mid A|_V \text{ is equicontinuous }\}$$

This is clearly an open subset of X. I claim that $A|_U$ is equicontinuous. Indeed, suppose $x \in U$ and $\epsilon > 0$. Then (by the definition of U) there is an open set V on which $A|_V$ is equicontinuous, such that $x \in V \subseteq U$. Thus (by equicontinuity on V) there is a set W open in V with $x \in W \subseteq V$ such that

$$f \in A, y \in W \Rightarrow |f(y) - f(x)| < \epsilon$$

Moreover, as W is open in V which is open in X, we see that W is open in X. This is precisely what is required for equicontinuity on U.

Given that $A|_V$ is equicontinuous, it is clear that V is the largest open set with the property that $A|_V$ is equicontinuous.

Consider the case $X = \mathbb{R}$ and $A = \{f_n \mid n \geq 2\}$ where $f_n(x) = x^n$. First, I claim that $A|_V$ is equicontinuous where $V = (-r, r)$ and $0 < r < 1$. Indeed, if $x, y \in (-r, r)$ where

$r < 1$ then

$$\begin{aligned} |x^n - y^n| &= |x-y||x^{n-1} + x^{n-2}y + \dots y^{n-1}| \\ &\leq |x-y|(|x^{n-1}| + \dots |y^{n-1}|) \\ &\leq |x-y|nr^{n-1} \\ &\leq 2|x-y|/(r^{-1}-1) \end{aligned}$$

This estimate is independent of n. It shows that A is equilipschitz and hence equicontinuous on $V = (-r, r)$. It follows that $A|_{(-1,1)}$ is equicontinuous (as $(-1,1) = \bigcup_{r<1}(-r, r)$).

Now suppose that $x \geq 1$. I claim that there is no neighbourhood V of x such that $A|_V$ is equicontinuous. To see this, suppose $y = x + u \geq x$. Then (by the binomial expansion)

$$y^n - x^n = (x+u)^n - x^n \geq nx^{n-1}u$$

Note that $nx^{n-1} \to \infty$ as $n \to \infty$. Thus, we can only have $y^n - x^n < \epsilon$ for all n if $u = 0$, i.e. if $y = x$. Thus, there is no neighbourhood W of x such that $y \in W$ implies $|f_n(x) - f_n(y)| < \epsilon$ for all n. In other words, A is not equicontinuous in any neighbourhood of x. A similar argument works if $x \leq -1$.

We can prove the inequality $nr^{n-1} \leq 2/(r^{-1} - 1)$ as follows. Write $\epsilon = r^{-1} - 1$ so $r = (1+\epsilon)^{-1}$. Then, by the binomial expansion, we have

$$(1+\epsilon)^{n-1} \geq (n-1)\epsilon$$

$$r^{n-1} = \frac{1}{(1+\epsilon)^{n-1}} \leq \frac{1}{(n-1)\epsilon}$$

Also, we assume $n \geq 2$ so $n/(n-1) \leq 2$. Thus

$$nr^{n-1} \leq \frac{n}{(n-1)\epsilon} \leq \frac{2}{\epsilon}$$

(16) Find examples of the following situations:
  (a) $X = \mathbb{Z}$ or $X = \{0\}$ or $X = \emptyset$.
  (b) $X = (0,1)$ or $X = \mathbb{Z}$.
  (c) $X = (0,1)$ or $X = \emptyset$.
  (d) $X = (-\infty, \pi) \cap \mathbb{Q}$ or $\{x \in \mathbb{Q} \mid x^2 < 2\}$.
  (e) $\{1/n \mid n \in \mathbb{Z}_+\} \cup \{0\}$ or the Cantor set.
  (f) $X = (-\infty, 0), Y = (0, \infty)$.
  (g) $U_n = (-2^{-n}, 2^{-n})$.

Consider a point $x \in X$. By Arzela-Ascoli, A is equicontinuous, so there is a neighbourhood U of x such that

$$y \in U,\ f \in A \quad \Rightarrow |f(x) - f(y)| < 1$$

I claim that $U = \{x\}$. Indeed, suppose not. Then there would be a point $y \neq x$ with $y \in U$. By Urysohn, we could choose a continuous function $f\colon X \to [0,1]$ with $f(x) = 0$ and $f(y) = 1$, violating the equicontinuity estimate. Thus $U = \{x\}$ is open for each $x \in X$, so the sets $\{x\}$ form an open cover of X. By compactness there is a finite subcover $\{\{x_1\}, \dots \{x_n\}\}$, so $X = \{x_1, \dots x_n\}$ is finite.

(17) (a)

$$\begin{aligned} ccA &= A \\ kkA &= kA \\ iiA &= iA \\ cicA &= kA \\ ckcA &= iA \end{aligned}$$

Further explanation is available on request.

  (b) As $U \subseteq A$ and U is open, we have $U \subseteq iA$. This implies that $A = kU \subseteq kiA$. On the other hand, we have $iA \subseteq A$ so $kiA \subseteq kA = kkU = kU = A$. Thus $kiA = A$.

(c) Applying the above to the case $U = iB$, we find that $kikiB = kiB$ for any B. Applying this in turn with $B = cC$ we get $cikikC = kikicC = kicC = cikC$ and thus $ikikC = ikC$.

(d) A typical set obtained from A by applying the operations i, k and c is something like $kkciccikkkciA$. We use the equations $ci = kc$ and $ck = ic$ to sweep the c's to the right, and then cancel them using $c^2 =$ identity. This leaves $kkkkiiiiA$. We then use $k^2 = k$ and $i^2 = i$ to eliminate repetitions, giving kiA. In the general case, we are left with a string of alternating i's and k's, followed either by A or by cA. If the string of i's and k's has length > 3, then we can use $kiki = ki$ or $ikik = ik$ to shorten it. This leaves us with 14 possibilities:

$$\begin{array}{rl} A & cA \\ iA & icA \\ kA & kcA \\ ikA & ikcA \\ kiA & kicA \\ kikA & kikcA \\ ikiA & ikicA \end{array}$$

The sets on the left are in some sense roughly the same size as A; they are at least bounded if A is bounded, for example. The ones on the right are roughly the same size as cA.

(e)

$$\begin{array}{rcl} A & = & A_0 \cup A_1 \cup A_2 \cup A_3 \\ A_0 & = & \mathbb{Q} \cap (0,1) \\ A_1 & = & [2,5] \setminus (\mathbb{Q} \cap (3,4)) \\ A_2 & = & \{6 + 1/n \mid n \in \mathbb{Z}_+\} \\ A_3 & = & [8,10] \setminus \{9 + 1/n \mid n \in \mathbb{Z}_+\} \end{array}$$

I'll draw a diagram by hand.

(19) (a)

$$\begin{array}{rcl} k(A \cup B) & = & kA \cup kB \\ i(A \cap B) & = & iA \cap iB \\ c(A \cap B) & = & cA \cup cB \\ A \cap (B \cup C) & = & (A \cap B) \cup (A \cap C) \\ bcA & = & bA \\ kbA & = & bA \end{array}$$

(b) Suppose A is closed. Then $bA = A \cap kcA \subseteq A$, so $ibA \subseteq iA$. On the other hand,

$$ibA \subseteq bA = A \cap kcA \subseteq kcA = ciA$$

Thus $ibA \subseteq iA \cap ciA = \emptyset$, so $ibA = \emptyset$. This implies that $bbA = kbA \cap kcbA = bA \cap cibA = bA$.

(c) For general A, we know that $B = bA$ is closed so by part (b) $b^2B = bB$, in other words $b^3A = b^2A$.

(d)

$$A = \mathbb{Q} \qquad bA = \mathbb{R} \qquad b^2A = \emptyset$$

(e) For any A we have $bA = kA \cap ciA = kA \setminus iA$, and it is always the case that $iA \subseteq A \subseteq kA$. From this we deduce that $bA = \emptyset$ iff $iA = A = kA$ iff A is both open and closed. If $X = \mathbb{R}$, this in turn implies that $A = \emptyset$ or $A = \mathbb{R}$ — this will be proved in class.

(20) Suppose X is Hausdorff, and $x \in X$. Suppose $y \neq x$. Then there are disjoint neighbourhoods U and V of x and y. In particular, $V \cap \{x\} = \emptyset$, so that y is not a closure point of $\{x\}$. It follows that $\{x\}$ is closed. If $F \subseteq X$ is finite, then it is a finite union of one point sets (which are closed) and so is closed.

The space Y described in the problem is a metric space, and thus Hausdorff. If $f, g \in Y$ and $f \neq g$ then $d = d(f,g) > 0$ and the sets $B(f, d/2)$ and $B(g, d/2)$ are disjoint neighbourhoods of f and g.

Given $0 < \delta \leq 1$, consider the function

$$f_\delta(x) = \begin{cases} 1 & \text{if } x \leq \delta/2 \\ 2 - 2t/\delta & \text{if } \delta/2 \leq x \leq \delta \\ 0 & \text{if } \delta \leq x \end{cases}$$

This is continuous, and satisfies $f_\delta \sim 1$ but

$$d(f_\delta, 0) = \int_0^1 |f_\delta(t)| dt = 3\delta/4 < \delta$$

Now write $X = Y/\sim$ and let $q \colon Y \to X$ be the usual quotient map. Suppose that $U \subseteq Y$ is a neighbourhood of $q(0)$. Then $U' = q^{-1}(U)$ is a neighbourhood of 0 in Y, so it contains a ball $B(0, \delta) = \{f \in Y \mid \int |f| < \delta\}$. In particular, $f_\delta \in U'$, so $q(f) \in U$. On the other hand, as $f \sim 1$ we have $q(1) = q(f)$. Thus, every neighbourhood of $q(0)$ contains $q(1)$. It clearly follows that we cannot have disjoint neighbourhoods of $q(0)$ and $q(1)$, so that X is not Hausdorff.

The set X is called the set of germs of continuous functions on $[0,1]$ at 0. It is a very useful set to consider, although the topology used in this question is of course perverse, immoral and contrary to reason.

(21) If i denotes the identity function on Y, then

$$f \times i \colon X \times Y \to Y \times Y$$

is continuous. As Y is Hausdorff, the diagonal $\Delta = \{(y,y) \mid y \in Y\}$ is closed in $Y \times Y$. The preimage $(f \times i)^{-1}(\Delta)$ is thus closed in $X \times Y$. This preimage is just

$$\{(x,y) \mid f(x) = i(y) = y\} = \{(x, f(x) \mid x \in X\} = \Gamma(f)$$

This shows that a continuous map to a Hausdorff space has closed graph. For a counterexample to the converse, define $f \colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1/x \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}$$

This is clearly not continuous. The graph can also be described as

$$\Gamma(f) = Y \cup \{(0,0)\} \qquad Y = \{(x,y) \mid xy = 1\}$$

The multiplication map $\mu \colon \mathbb{R}^2 \to \mathbb{R}$ is continuous, so $Y = \mu^{-1}\{1\}$ is closed, so $\Gamma(f)$ is closed.

(22) First, note that $f(f(z)) = z$ for all z, so f has inverse f and is a bijection.

We need only check continuity of f at 0 and ∞ — we are allowed to assume it elsewhere. The basic neighbourhoods of $f(0) = \infty$ are the sets

$$V_R = \{z \in \mathbb{C} \mid |z| > R\} \cup \{\infty\}$$

The preimage is

$$f^{-1}(V_R) = \{w \in \mathbb{C} \mid |w| < R^{-1}\}$$

which is certainly a neighbourhood of 0. Similarly, the preimage of a basic neighbourhood $\{z \mid |z| < \epsilon\}$ of 0 is the neighbourhood $V_{1/\epsilon}$ of ∞. It follows that f is continuous, and thus that $f^{-1} = f$ is continuous, and thus that f is a homeomorphism.

Next, we define two open subsets of S^3:

$$U_0 = \{(z,w) \in S^3 \mid w \neq 0\}$$

$$U_\infty = \{(z,w) \in S^3 \mid z \neq 0\}$$

Because

$$(z,w) \in S^3 \Rightarrow |z|^2 + |w|^2 = 1 \Rightarrow z \neq 0 \text{ or } w \neq 0$$

we see that $U_0 \cup U_1 = S^3$. It is thus enough to check that the restrictions $g|_{U_0}$ and $g|_{U_1}$ are continuous. If we write 1 for the identity map of $\mathbb{C}$ and suppress mention of inclusion maps, we have

$$g|_{U_0} = m \circ (1 \times h)$$
$$g|_{U_1} = f \circ m \circ (h \times 1)$$

which shows that the restrictions of g are continuous as required.

(23)

(24) First, the map $i_x \colon y \mapsto (x,y)$ is clearly continuous, so $k^\#(x) = k \circ i_x$ is continuous.

Next, suppose given $x \in [0,1]$ and $\epsilon > 0$. Write

$$U = \{(y,z) \in [0,1]^2 \mid |k(y,z) - k(x,z)| < \epsilon\}$$

Clearly $\{x\} \times [0,1] \subseteq U$. Thus, by the tube lemma, there is a neighbourhood V of x such that $V \times [0,1] \subseteq U$. Thus, for $y \in V$ and any z we have

$$|k^\#(y)(z) - k^\#(x)(z)| = |k(y,z) - k(x,z)| < \epsilon$$

so $\|k^\#(x) - k^\#(y)\| < \epsilon$. This shows that $k^\#$ is continuous.

Now define

$$K \colon C[0,1] \to C[0,1]$$
$$(Ku)(x) = \int_0^1 k(x,y)u(y)\mathrm{d}y$$

We have

$$\begin{aligned}
|(Ku)(x) - (Ku)(y)| &= \left|\int_0^1 k(x,z)u(z) - k(y,z)u(z)\mathrm{d}z\right| \\
&\leq \int_0^1 |k^\#(x)(z) - k^\#(y)(z)||u(z)|\mathrm{d}z \\
&\leq \|k^\#(x) - k^\#(y)\|\|u\|
\end{aligned}$$

Fix $u \in C[0,1]$. Given x and $\epsilon > 0$ we can find a neighbourhood V of x such that

$$y \in V \Rightarrow \|k^\#(x) - k^\#(y)\| < \epsilon/\|u\| \Rightarrow |(Ku)(x) - (Ku)(y)| < \epsilon$$

This shows that Ku is continuous.

Now let u vary. Write

$$\|k\| = \max\{|k(x,y)| \mid x,y \in [0,1]\}$$

which is well-defined and finite because $[0,1]^2$ is compact and k is continuous. We have

$$|(Ku)(x) - (Kv)(x)| \leq \int_0^1 |k(x,y)||u(y) - v(y)|\mathrm{d}y \leq \|k\|\|u - v\|$$

This shows that K is Lipschitz and thus continuous.

(25)

$$\begin{aligned}
|Kf(t) - Kg(t)| &= \left|\int_0^1 (f(s) - g(s))(s+t)^2\mathrm{d}s\right| \\
&\leq \int_0^1 |f(s) - g(s)|(s+t)^2\mathrm{d}s \\
&\leq \int_0^1 4|f(s) - g(s)|\mathrm{d}s \\
&\leq 4\|f - g\|
\end{aligned}$$

This shows that $\|Kf - Kg\| \leq 4\|f - g\|$, so K is lipschitz and thus continuous.

Next, write

$$B = \{f \in X \quad | \quad \|f\| \leq 1\}$$

We need to prove that $V \cap B$ is closed, bounded and equicontinuous, so we can apply the Arzela-Ascoli theorem to show that it is compact.

The set B is clearly closed and bounded. The set V is also closed, because it is the preimage of the closed set $\{0\}$ under the continuous map $K - \text{id}$. Thus $V \cap B$ is closed and bounded.

Next, observe that

$$|(s+t)^2 - (s+t')^2| = |2t + t^2 - 2t' - t'^2| = |2 + t + t'||t - t'| \leq 4|t - t'|$$

so

$$|Kf(t) - Kf(t')| \leq \int_0^1 |(s+t)^2 - (s+t')^2||f(s)|\mathrm{d}s \leq 4|t-t'| \int_0^1 |f(s)|\mathrm{d}s \leq 4|t-t'|\|f\|$$

In particular, for $f \in B$ we have

$$|Kf(t) - Kf(t')| \leq 4|t - t'|$$

Thus for $f \in V \cap B$ (so $f = Kf$) we have

$$|f(t) - f(t')| \leq 4|t - t'|$$

This shows that the family $V \cap B$ is equilipschitz and thus equicontinuous.

(26) No. Let X be the discrete space $\mathbb{N}$, which is locally connected ($\{n\}$ is a connected neighbourhood of n contained in every neighbourhood of n). Let Y be $\{1/n \mid n \in \mathbb{Z}_+\} \cup \{0\}$. The map $f \colon X \to Y$ defined by

$$f(n) = \begin{cases} 0 & \text{if } n = 0 \\ 1/n & \text{if } n > 0 \end{cases}$$

is surjective and continuous (trivially, because $\mathbb{N}$ is discrete). The point $0 \in Y$ has no connected neighbourhoods, so Y is not locally connected.

(27) (a) Suppose that $U \subseteq \mathbb{R}^n$ is open. Any subspace of a Hausdorff space is Hausdorff, and any subspace of a second countable space is second countable, so the first two conditions are trivial. Suppose $x \in U$, so there is some basic neighbourhood

$$x \in V = (x_1 - \epsilon, x_1 + \epsilon) \times \ldots \times (x_n - \epsilon, x_n + \epsilon) \subseteq U$$

There is then a homeomorphism $V \to \mathbb{R}^n$ constructed in the obvious way from the homeomorphism

$$(-\epsilon, \epsilon) \to \mathbb{R} \qquad\qquad y \mapsto y/(\epsilon^2 - y^2)$$

Now suppose that M is a manifold and $N \subseteq M$ is open. Then N is again Hausdorff and second countable for the same reasons. Suppose $x \in N$. Choose a neighbourhood U of x in M homeomorphic to $\mathbb{R}^n$, so $N \cap U$ is homeomorphic to an open subset of $\mathbb{R}^n$ and thus contains a neighbourhood V of x homeomorphic to all of $\mathbb{R}^n$. Thus N is a manifold.

(b) Suppose that M is a manifold and $N \subseteq M$ is a component. First, note that M is locally euclidean and hence locally path-connected and hence locally connected. It follows that the components (such as N) are open, so they are manifolds by the last part. Moreover, N is connected because it is a component. However, a connected and locally path connected space is path connected. Thus N is a path connected manifold.

(c) Suppose $x \in M$. Choose a neighbourhood U of x and a homeomorphism $f \colon \mathbb{R}^n \to U$. We may assume $f(0) = x$ (else consider $g(u) = f(a + u)$ where $f(a) = x$). Write $B = \{x \in \mathbb{R}^n \mid \|x\| < 1\}$ and $W = f(B)$. As $f \colon \mathbb{R}^n \to U$ is a homeomorphism, W is open in U and hence open in M (as U is).

Note that $\overline{B}$ is compact so $f(\overline{B})$ is compact and thus closed. It also contains $W = f(B)$, so $f(\overline{B}) \supseteq \overline{f(B)} = \overline{W}$. This shows that $\overline{W}$ is compact. Thus W is a precompact neighbourhood of x. Thus M is locally compact.
It follows that M is locally compact, Hausdorff and second countable. Such a space is always paracompact.

(d) The space $\mathbb{Q}$ is Hausdorff and second countable but not locally euclidean (because not locally connected, for example).
Consider an uncountable set X with the discrete topology. Then X is Hausdorff and locally homeomorphic to $\mathbb{R}^0$, but is not second countable. If you don't like $n = 0$, take $X \times \mathbb{R}$.
Finally, consider the "line with two zeros":

$$Y = \mathbb{R} \times \{-1, 1\}/\sim$$

$$(s, -1) \sim (s, 1) \text{ unless } s = 0$$

The points consist of equivalence classes $[(s, -1)] = [(s, 1)]$ for $s \neq 0$ (which I'll just write as s) and the points $0_+ = [(0, 1)]$ and $0_- = [(0, -1)]$. I proved in class that Y is not Hausdorff (because 0_+ and 0_- cannot be separated). However, you can check that

$$q\colon (a, b) \times \{1\} \to q((a, b) \times \{1\})$$

and

$$q\colon (a, b) \times \{-1\} \to q((a, b) \times \{-1\})$$

are homeomorphisms and that the images are open in Y. It follows easily from this that Y is second countable (take $a, b \in \mathbb{Q}$) and locally euclidean.

(28) (a) Given a matrix $M = \{m_{kl}\}$, we write $\pi_{kl}(M) = m_{kl}$. We topologise the space M_n as a product of n^2 copies of $\mathbb{R}$, so the projections $\pi_{kl}\colon M_n \to \mathbb{R}$ are continuous. As sums, products and constant multiples of continuous functions are again so, it follows that any function we can build from the projections by such steps will be continuous. The determinant is certainly such a function (in the case $n = 2$, we have $\det = \pi_{11}\pi_{22} - \pi_{12}\pi_{21}$, for example).

(b) A matrix M is invertible iff $\det(M) \neq 0$. We define

$$GL_n^+ = \{M \mid \det(M) > 0\}$$

$$GL_n^- = \{M \mid \det(M) < 0\}$$

These sets are open because det is continuous, and $GL^+ \cup GL^- = GL$. Moreover, both are nonempty — the identity matrix is in GL^+ and the matrix which is like the identity but with a single 1 changed to a -1 is in GL^-.
Here is a sketch proof that the complex matrix group $GL_n(\mathbb{C})$ is connected; a similar proof can be given that GL_n^+ is connected. Suppose that M is an $n \times n$ invertible complex matrix. Suppose first for simplicity that M has n distinct eigenvalues, which are neccessarily nonzero as $\det(M) \neq 0$. They can thus be written as e^{α_k} for $k = 1 \dots n$. By the usual results of linear algebra, M can be written as $A^{-1}D(1)A$ for some matrix A, where $D(t)$ is the diagonal matrix with entries $e^{t\alpha_k}$. The matrices $A^{-1}D(t)A$ form a path between the identity matrix and M as t varies from 0 to 1. If M does not have n distinct eigenvalues, then we can still find a matrix D in Jordan canonical form and another matrix A with $M = A^{-1}DA$. By looking at the definition of a Jordan canonical form, we see that we can change D (and hence M) by an arbitarily small amount and make it have distinct eigenvalues. This shows that the set of matrices with distinct nonzero eigenvalues is dense in $GL_n(\mathbb{C})$. Using the fact that the closure of a connected set is connected, we can complete the proof.

(c) Suppose that $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SO_2$. Then

$$\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) = MM^T = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} a^2+b^2 & ac+bd \\ ac+bd & c^2+d^2 \end{smallmatrix}\right)$$

and

$$\det(M) = ad - bc = 1$$

Using the resulting relations $a^2 + b^2 = 1$ and $c^2 + d^2 = 1$ we get

$$(a-d)^2 + (b+c)^2 = a^2 + d^2 + b^2 + c^2 - 2(ad - bc) = 0$$

This implies that $d = a$ and $c = -b$, so $M = \left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right)$. We can now use the criteria for maps to products and subspaces to see that the map $SO_2 \to S^1 = \{(x, y) \mid x^2 + y^2 = 1\}$ sending M to (a, b) is a homeomorphism.

(d) The map $f : M_n \to M_n$ defined by $f(A) = A^T A$ is continuous. Indeed, the composites $\pi_{kl} \circ f$ are given by

$$(\pi_{kl} \circ f)(M) = \sum_i m_{ik} m_{il} = \left(\sum_i \pi_{ik} \pi_{il} \right)(M)$$

This is continuous by the usual argument about continuity of algebraic operations. By the criterion for maps to a product space, f itself is continuous. Thus $O_n = f^{-1}\{I\}$ is closed. Moreover, by looking at the diagonal entries in $A^T A$ we see that $\sum_i m_{ik}^2 = 1$ so $|m_{ik}| \leq 1$ for all i and k. Thus O_n is compact.

(e) Suppose $\|\underline{x}\| \leq 1$. Write $a = \max\{A_{kl}\}$ and $\underline{y} = A\underline{x}$, so

$$y_k = \sum_k A_{kl} x_l$$

$$|y_k| \leq \sum_k |A_{kl}||x_l| \leq na \max\{x_l\} \leq na$$

$$\|\underline{y}\|^2 = \sum y_k^2 \leq n(na)^2 = n^3 a^2$$

$$\|\underline{y}\| \leq n^{3/2} a$$

so $\|A\|_{op} \leq n^{3/2} \|A\|_\infty$. This is all terribly crude, but it will do.

(f) Given two matrices A and B, we have

$$\|(A + B)\underline{x}\| = \|A\underline{x} + B\underline{x}\| \leq \|A\underline{x}\| + \|B\underline{x}\|$$

Taking the least upper bound as $\underline{x}$ ranges over vectors of norm at most one, we find that

$$\|A + B\|_{op} \leq \|A\|_{op} + \|B\|_{op}$$

so

$$\|A\|_{op} = \|A - B + B\|_{op} \leq \|A - B\|_{op} + \|B\|_{op}$$

hence (using also the inequality with A and B exchanged)

$$|\; \|A\|_{op} - \|B\|_{op} \;| \leq \|A - B\|_{op} \leq n^{3/2} \|A - B\|_\infty$$

This shows that $\|A\|_{op}$ is a continuous function of A.

(g) First, recall a statement of Rouché's theorem: if f and g are entire analytic functions and $|f(z) - g(z)| < |f(z)|$ for all z on some closed contour Γ, then f and g have the same number of zeros (counted by multiplicity) inside Γ.

Let $p(M, t)$ denote the characteristic polynomial $\det(M - tI)$ of M. The function from $M_n \times \mathbb{C}$ to $\mathbb{C}$ sending (M, z) to $p(M, z)$ is continuous. Suppose that the eigenvalues of M are $\lambda_1, \ldots \lambda_m$, with multiplicities $v_1, \ldots v_m$. Given $\epsilon > 0$, construct a contour Γ consisting of small circles of radius less than epsilon, with one circle surrounding each eigenvalue and not touching any other circle or winding around any other eigenvalue. Let U denote the set of pairs (N, z) such that $|p(N, z) - p(M, z)| < |p(M, z)|$. This is open and contains $\{M\} \times \Gamma$. Note that Γ is compact. By "Step 1" on p.168, there is a neighbourhood V of M such that $V \times \gamma \subseteq U$. This implies via Rouché's theorem that for each $N \in V$, the matrix N has the same number of eigenvalues inside each circle of Γ as M has. This implies in turn that each eigenvalue of N lies within distance ϵ of an eigenvalue of M, and *vice versa*. It follows that $|r(M) - r(N)| < \epsilon$.

(29) Suppose that $f(z) = a_0 + \dots a_n z^n$ is a nonconstant complex polynomial of degree n (so $n > 0$ and $a_n \neq 0$).
  (a) Let $D \subseteq \mathbb{C}$ be open. The collection of those balls
$$B(z, \epsilon) = \{w \in \mathbb{C} \mid \quad |z - w| < \epsilon\}$$
which happen to be contained in D form a basis for the topology on D. These balls are all convex hence path connected hence connected. Thus D is locally connected. This implies that the components of D are open in D (and hence open in $\mathbb{C}$, although this is not immediately relevant). The complement of a component is the union of all the other components, hence open in D. Thus each component is also closed in D.
  (b) K closed and bounded $\Leftrightarrow$ K compact $\Rightarrow$ $f(K)$ compact $\Leftrightarrow$ $f(K)$ closed and bounded.
  (c) Write
$$K = \max(1, (2|a_n|)^{1/n}, 2|a_n|^{-1} \sum_{k<n} |a_k|)$$
For $|z| > \max(1, 2|a_n|^{-1} \sum_{k<n} |a_k|)$ we have
$$|f(z)| \geq |a_n||z|^n - \sum_{k<n} |a_k||z|^k \geq |a_n||z|^n/2$$
and so for $|z| > K$ we have $|f(z)| > 1$, as required.
On the other hand, by part (b) we know that $f(\{z \mid \quad |z| \leq K\})$ is bounded, so that there exists $L > 0$ such that $|z| \leq K \Rightarrow |f(z)| \leq L$ and thus
$$|f(z)| > L \Rightarrow |z| > K$$
again as required.
  (d) By the previous part, V contains the set $E = \{z \in \mathbb{C} \mid \quad |z| > K\}$. As E is connected, it is contained in one of the components of V. As E is unbounded, this means that V has at least one unbounded component.
Next, suppose that W and W' are unbounded components of V. Then $W \cap E \neq \emptyset$ because W is unbounded. As E is connected, this implies $E \subseteq W$. Similarly $E \subseteq W'$, and thus $W \cap W' \supseteq E \neq \emptyset$. As the components of V form a partition of V, this implies that $W = W'$. Thus V has precisely one unbounded component.
  (e) Suppose that W is a bounded component of V. Then by (a), W is open and closed in V.
W is contained in the closed set $f^{-1}\left(\overline{f(W)}\right)$, so $\overline{W}$ is also contained in this set, so $f(\overline{W}) \subseteq \overline{f(W)}$.

On the other hand, $\overline{W}$ is compact, so $f(\overline{W})$ is closed and contains $f(W)$, so $\overline{f(W)} \subseteq f(\overline{W})$. Thus
$$\overline{f(W)} = f(\overline{W})$$
You can check that
$$f(\overline{W}) \cap U = f(\overline{W} \cap f^{-1}(U)) = f(\overline{W} \cap V)$$
(this is purely set theoretic, not involving any topology). On the other hand, W is closed in V so $\overline{W} \cap V = W$. We conclude that
$$\overline{f(W)} \cap U = f(W)$$
as required.
This shows that $f(W)$ is closed in U. Also, W is open in V and thus in $\mathbb{C}$, and $f \colon \mathbb{C} \to \mathbb{C}$ is an open mapping, so $f(W)$ is open in U. On the other hand, $U = \{z \mid \quad |z| > 1\}$ is easily seen to be connected, so the open and closed set $f(W)$ can only be $\emptyset$ or U. The former is excluded because W is a component and therefore nonempty. Thus $f(W) = U$. However, $f(W) \subseteq f(\overline{W})$ which we have seen is bounded. This contradiction shows that V cannot have any bounded components. Combining this with (d), we see that V has precisely one component and is therefore connected.

(f) Write $V_n = \{z \mid \quad |f_n(z)| > 2\}$, so that $M^c = \bigcup_n V_n$. By the above, V_n is connected for each n. Moreover, there are constants K_n such that $|z| > K_n \Rightarrow z \in V_n$. From this we see that $V_n \cap V_m \neq \emptyset$ for each n and m. Thus M^c is the union of a family of connected sets, any pair of which intersect, so it is itself connected.

(30) Given a complex number c, define $q_c(z) = z^2 + c$ and

$$f_n(c) = q_c(q_c(q_c \dots (0) \dots))$$

where q_c is applied n times. In other words:

$$f_0(c) = 0$$
$$f_{n+1}(c) = q_c(f_n(c)) = f_n(c)^2 + c$$

In particular:

$$\begin{aligned} f_1(c) &= c \\ f_2(c) &= c^2 + c \\ f_3(c) &= c^4 + 2c^3 + c^2 + c \end{aligned}$$

We see by induction using the formula $f_{n+1}(c) = f_n(c)^2 + c$ that f_n is continuous for all n. It is also true that the function

$$g_{c,n}(z) = q_c(q_c(\dots(z)\dots))$$

is a continuous function of z, but this is a different thing.

The Mandelbrot set M is defined as

$$M = \{c \in \mathbb{C} \mid |f_n(c)| \leq 2 \text{ for all } n\}$$

This is bounded because if $c \in M$ then (by the case $n = 1$) we have $|c| \leq 2$. If we write $B = \{z \mid |z| \leq 2\}$ then M can also be described as

$$M = \bigcap_n f_n^{-1}(B)$$

As B is closed and f_n is continuous, we see that $f_n^{-1}(B)$ is closed. Thus M is an intersection of closed sets and hence closed.

(31)

(32) (a) Yes.

(b) No. The triangle inequality M2 fails for $x = -1$, $y = 0$, $z = 1$ for example.

(c) Yes. To prove this, it helps to show first that

$$\overline{d}(x, y) = \min(|x - y|, 1)$$

gives a metric on $\mathbb{R}$ (in fact, it induces the same topology as the usual metric). This is theorem 9.1 in the book.

(d) This is certainly not a metric space, as we have

$$d((1, 0), (0, 1)) = 0 \text{ but } (1, 0) \neq (0, 1)$$

contrary to axiom M3. It is not even a pseudometric space, as the triangle inequality fails for $x = (0, 0)$, $y = (1, 0)$ and $z = (1, 1)$.

(e) Yes.

(f) No. The triangle inequality fails for $x = 0$, $y = 1$ and $z = 3$.

(33) Let e_k be the k'th basis vector $(0, \dots, 1, \dots 0)$ and write $a_k = \|e_k\|$. Then

$$\|x\| = \|\sum x_k e_k\| \leq \sum |x_k| \|e_k\| = \sum |x_k| a_k$$

$$|\, \|x\| - \|y\|| \leq \|x - y\| \leq \sum_k |x_k - y_k| a_k$$

It follows that $|n(y) - n(x)| < \epsilon$ provided that

$$y \in \prod_k (x_k - \delta, x_k + \delta)$$

where
$$\delta = \epsilon / \sum_k a_k$$
This shows that n is continuous for the product topology.

Next, recall that the sphere $S^{n-1} = \{x \mid \|x\|_2 = 1\}$ is compact in the product topology. It follows that n is bounded on S^{n-1}, say $n(x) \leq K$ for $x \in S^{n-1}$. More generally, if $x \neq 0$ then $x/\|x\|_2 \in S^{n-1}$ so
$$n(x)/\|x\|_2 = n(x/\|x\|_2) \leq K$$
so
$$n(x) \leq K\|x\|_2$$
This also holds for $x = 0$, of course.

Moreover, $n(x) > 0$ for $x \in S^{n-1}$ (because $\|x\| = 0 \Rightarrow x = 0$) so $1/n(x)$ is also continuous and thus bounded on S^{n-1}. We deduce that
$$n(x)^{-1} \leq k^{-1} \text{ for } x \in S^{n-1}$$
$$k\|x\|_2 \leq n(x) \text{ for all } x$$
It follows that the identity maps
$$(\mathbb{R}^n, d) \to (\mathbb{R}^n, d_2) \to (\mathbb{R}^n, d)$$
are both Lipschitz and therefore continuous, in other words that the two topologies are the same.

(34) Suppose X is compact Hausdorff, and K, L are disjoint closed subsets. For $x \in K$ write
$$\mathcal{V}_x = \{V \in \tau \mid x \notin \overline{V}\}$$
Suppose $y \in L$, so $y \neq x$. As X is Hausdorff, there are disjoint neighbourhoods U and V of x and y. As $U \cap V = \emptyset$, we see that $x \notin \overline{V}$ and thus $V \in \mathcal{V}_x$. Note also that $y \in V$; this shows that
$$L \subseteq \bigcup_{V \in \mathcal{V}_x} V$$
It follows that there are sets $V_1, \dots V_r \in \mathcal{V}_x$ such that
$$L \subseteq V = \bigcup_{k=1}^{r} V_k$$
moreover
$$x \in U = \bigcap_{k=1}^{r} (X \setminus \overline{V_k}) = X \setminus \bigcup \overline{V_k} = X \setminus \overline{V}$$
Also, if $y \in L$ then V is a neighbourhood of y and $V \cap U = \emptyset$; thus $\overline{U} \cap L = \emptyset$.

Next, we define
$$\mathcal{U} = \{U \in \tau \mid \overline{U} \cap L = \emptyset\}$$
The above shows that $\mathcal{U}$ covers K, so
$$K \subseteq U = \bigcup_{l=1}^{s} U_l \qquad U_l \in \mathcal{U}$$
say. We can then take
$$V = \bigcap_{l=1}^{s} (X \setminus \overline{U_l})$$
We find that $K \subseteq U$, $L \subseteq V$ and $U \cap V = \emptyset$ as required.

(35) In these cases Y is open in X:
   (a) $X = \mathbb{R}$, $\quad Y = \mathbb{R} \setminus \mathbb{Z} = \{x \in \mathbb{R} \mid x \notin \mathbb{Z}\}$
   (b) $X = \mathbb{Q}$, $\quad Y = \mathbb{Q} \cap [-\sqrt{2}, \sqrt{2}]$
   (c) $X = \{1/n \mid n \in \mathbb{Z}_+\}$, $\quad Y = \{1/(n+1) \mid n \in \mathbb{Z}_+\}$
   (d) $X = [0,1] \cup [2,3]$, $\quad Y = [0,1]$

Note that in the second case $Y = (-\sqrt{2}, \sqrt{2}) \cap \mathbb{Q}$.

In these cases Y is not open in X:

(a) $X = [-1, 1]$, $\quad Y = [0, 1]$

(b) $X = \mathbb{R}$, $\quad Y = \{x \in \mathbb{R} \mid x \neq 1/n \text{ for any } n \in \mathbb{Z}_+\}$

In both cases 0 is a non-interior point of Y.

(36) (a) No. If the range of a function $f : \mathbb{R} \to \mathbb{R}$ is finite then $f(F)$ is finite and hence closed for every closed set $F \subseteq \mathbb{R}$ (indeed, for every subset $F \subseteq \mathbb{R}$ whatsoever). However, f certainly need not be continuous; take for example $f(x) = 0$ for $x < 0$ and $f(x) = 1$ for $x \geq 0$.

(b) No. For example if $f(x) = \sin(x)$ then $f(\mathbb{R})$ is not open, so f is not open.

(c) Yes. If $f : X \to Y$ is a homeomorphism, then $g = f^{-1} : Y \to X$ is continuous. Thus, if $F \subseteq X$ is closed then $f(F) = g^{-1}(F)$ is closed in Y.

(d) No. If $f(x) = e^{-x}$ then $f(\mathbb{R}) = (0, \infty)$ is not closed in $\mathbb{R}$, so $f : \mathbb{R} \to \mathbb{R}$ is not a closed mapping.

(e) Yes. Suppose that X and Y are compact Hausdorff and that $f : X \to Y$ is continuous. Then, if $F \subseteq X$ is closed then it is compact, so $f(F)$ is compact. As a compact subset of a Hausdorff space, $f(F)$ must be closed.

(37) The metric is derived from the norm

$$\|u\| = \|u\|_\infty = \sup\{|u(x)| \mid x \in X\}$$

Thus, if $u \in C(Y)$ then

$$\begin{aligned} \|f^*(u)\| &= \sup\{|f^*(u)(x)| \mid x \in X\} \\ &= \sup\{|u(f(x))| \mid x \in X\} \\ &\leq \sup\{|u(y)| \mid y \in Y\} = \|u\| \end{aligned}$$

Noting also that $f^*(u - v) = f^*(u) - f^*(v)$, we find that $d(f^*(u), f^*(v)) \leq d(u, v)$. This implies that f^* is continuous.

It is also easy to see that the norm function

$$n : C(X) \to \mathbb{R} \qquad n(u) = \|u\|$$

is continuous. This follows from the reversed triangle inequality:

$$|n(u) - n(v)| \leq d(u, v)$$

Now consider $Y = \{(x, x') \in X^2 \mid d(x, x') < \epsilon\}$. There are two continuous projection maps $\pi_0, \pi_1 : Y \to X$. We have

$$\text{osc}_\epsilon(u) = n(\pi_0^*(u) - \pi_1^*(u))$$

which shows that osc_ϵ is continuous.

We next want to show that $\bigcup_{\epsilon>0} U(\epsilon, \delta) = C(X)$. Consider $u \in C(X)$; we need to find $\epsilon > 0$ such that $\text{osc}_\epsilon(u) < \delta$. This just means that u is uniformly continuous. A proof in the spirit of this problem is as follows. Write

$$K = \{(x, x') \in X^2 \mid \ |u(x) - u(x')| \geq \delta\}$$

The image $d(K)$ under the distance map $d : X^2 \to \mathbb{R}$ is compact and does not contain 0, so $d(K) \cap [0, \epsilon) = \emptyset$ for some $\epsilon > 0$. Thus $d(x, x') < \epsilon$ implies $|u(x) - u(x')| < \delta$ as required.

(38) (a) First, $\mathbb{Z}_p$ is a metric space and therefore Hausdorff. The subspace $\mathbb{Z}$ is dense, so for compactness it is sufficient to prove that $\mathbb{Z}$ is totally bounded, in other words that it has a finite ϵ-net for every $\epsilon > 0$. For this, choose n such that $p^{-n} < \epsilon$. Any integer m is congruent modulo p^n to a number l with $0 \leq l < p^n$:

$$m = kp^n + l \qquad k \in \mathbb{Z} \quad 0 \leq l < p^n$$

Thus $d(m, l) = |kp^n| \leq p^{-n} < \epsilon$. This shows that $\{0, 1, 2, \ldots p^n - 1\}$ is an ϵ-net.

(b) First, we must prove that $\sim_k$ is an equivalence relation. This is trivial once we remark that

$$n \sim_k m \;\Leftrightarrow\; n - m \text{ is divisible by } p^k$$

Consider the space $\mathbb{Z}/\sim_k = \mathbb{Z}/p^k$. If $a = [n]_k \in \mathbb{Z}/p^k$ then

$$q_k^{-1}\{a\} = \{m \in \mathbb{Z} \mid m \sim_k n\} = B(n, p^{1-k})$$

This shows that $q_k^{-1}\{a\}$ is open, so $\{a\}$ is open in the quotient topology. As the points are open, the space is discrete. By the argument of the previous question,

$$\mathbb{Z}/p^k = \{[0]_k, \ldots [p^k - 1]_k\}$$

which is finite.

(c) The apparent problem with the definition $r_k([n]_k) = [n]_{k-1}$ is that we might have $a = [n]_k = [m]_k \in \mathbb{Z}/p^k$ and then we would have two potentially different definitions $r_k(a) = [n]_{k-1}$ and $r_k(a) = [m]_{k-1}$. Of course, they are not really different:

$$[n]_k = [m]_k \Leftrightarrow v(n-m) \geq k \Rightarrow v(n-m) \geq k-1 \Leftrightarrow [n]_{k-1} = [m]_{k-1}$$

(d) We next consider the space

$$X = \{a \mid \forall k > 0 \quad r_k(a_k) = a_{k-1}\} \subset \prod_{k \in \mathbb{N}} \mathbb{Z}/p^k$$

If $a, b \in X$ and $a \neq b$ then for some k we have $a_k \neq b_k$. Thus $\pi_k^{-1}\{a_k\} \cap X$ and $\pi_k^{-1}\{b_k\} \cap X$ are disjoint neighbourhoods of a and b. Thus X is Hausdorff. Alternatively, we could just quote the fact that products and subspaces of Hausdorff spaces are Hausdorff.
The infinite product is compact by Tychonov, so we need only show that the subspace X is closed. Suppose $a \in \prod_k \mathbb{Z}/p^k$ but $a \notin X$. Then $r_k(a_k) \neq a_{k-1}$ for some k. Write

$$U = \pi_k^{-1}\{a_k\} \cap \pi_{k-1}^{-1}\{a_{k-1}\}$$

This is a neighbourhood of a. Moreover, if $b \in U$ then

$$r_k(b_k) = r_k(a_k) \neq a_{k-1} = b_{k-1}$$

so $b \notin X$. Thus X is closed in a compact space and thus compact.

(e) Next, we consider the sets

$$U_k(c) = \pi_k^{-1}\{c\} \cap X = \{a \in X \mid a_k = c\}$$

As the sets $\pi_k^{-1}\{c\}$ form (by definition) a subbasis for the product topology, it is immediate that the sets $U_k(c)$ form a subbasis for the topology on X as a subspace of the product. The claim is that they are not merely a subbasis but a basis.
Consider $V = U_k(c) \cap U_l(b)$, with $k \leq l$ say. Note that $a \in V$ iff $a_k = c$ and $a_l = b$. For $a \in X$ we have

$$a_k = r_{k+1}(a_{k+1}) = r_{k+1} r_{k+2} \ldots r_l(a_l)$$

This shows that $V = U_l(b)$ if $r_{k+1} r_{k+2} \ldots r_l(b) = c$ and $V = \emptyset$ otherwise. Thus, a finite intersection of sets in our subbasis either lies again in the subbasis or is empty. It follows easily that the subbasis is really a basis.

(f) Suppose $a, b \in X$ and that $a_k = b_k$. Working downwards using $a_{k-1} = r_k(a_k)$ etc. we deduce that $a_l = b_l$ for all $l \leq k$.
Suppose that $a \in X$ and $p^{-n} < \epsilon \leq p^{1-n}$. It is immediate from the definitions and the above remark that

$$b \in B(a, \epsilon) < \epsilon \Leftrightarrow \exists m \geq n \; b_m = a_m \Leftrightarrow b_n = a_n \Leftrightarrow b \in U_n(a_n)$$

This shows that $B(a,\epsilon) = U_n(a_n)$, and thus that the basis constructed previously is precisely the set of open balls. It follows that the metric topology is the same as the previous one.

(g) Define $f\colon \mathbb{Z} \to X$ by $f(n) = ([n]_k)_{k\in\mathbb{N}}$. We then have

$$v(f(n)) = v(n)$$
$$|f(n)| = |n|$$
$$d(f(n), f(m)) = d(n,m)$$

so f is an isometric embedding. Moreover, X is compact and thus complete. By general results about completion, there is an isometric embedding $\tilde{f}\colon \mathbb{Z}_p \to X$ extending f.

Consider a basic open set $V = U_k(c)$, where $c = [n]_k \in \mathbb{Z}/p^k$ say. Then $f(n) \in V$. It follows that $f(\mathbb{Z})$ is dense in $\mathbb{Z}_p$. A fortiori, $\tilde{f}(\mathbb{Z}_p)$ is dense. On the other hand, $\mathbb{Z}_p$ is compact so $\tilde{f}(\mathbb{Z}_p)$ is compact and thus closed. This means that $\tilde{f}(\mathbb{Z}_p) = X$, so $\tilde{f}$ is surjective. As it is isometric, it is injective:

$$x \neq y \Leftrightarrow 0 \neq d(x,y) = d(\tilde{f}(x), \tilde{f}(y)) \Leftrightarrow \tilde{f}(x) \neq \tilde{f}(y)$$

it also follows easily that the inverse is an isometry:

$$d(\tilde{f}^{-1}(x), \tilde{f}^{-1}(y)) = d(\tilde{f}\tilde{f}^{-1}(x), \tilde{f}\tilde{f}^{-1}(y)) = d(x,y)$$

It follows that the inverse is continuous and thus $\tilde{f}$ is a homeomorphism.

(39) The set $V = (0,\epsilon)\times[-\pi,\pi]$ is connected, because intervals in $\mathbb{R}$ are connected and products of connected sets are connected.

The set U is the image of V under the continuous map $(r,\theta) \mapsto a + (r\cos\theta, r\sin\theta)$. Continuous images of connected sets are connected, so U is connected.

Suppose that $X \subseteq \mathbb{R}^2$ is connected and $a \in \operatorname{int}(X)$. Write $Y = X \setminus \{a\}$, and choose $\epsilon > 0$ such that

$$U' = \{b \mid d(a,b) < \epsilon\} \subseteq X$$

so

$$U = \{b \mid 0 < d(a,b) < \epsilon\} \subseteq Y$$

Suppose that (A,B) is a separation of Y, so A and B are open in Y and are disjoint, and $A \cup B = Y$. Then $(A\cap U, B\cap U)$ is a separation of the connected set U, hence is trivial. Without loss of generality, we may assume $U \subseteq A$ and $U \cap B = \emptyset$. Write $A' = A \cup U' = A \cup \{a\}$. Note that Y is open in X and thus A, B and A' are open in X. Moreover, $A' \cap B = \emptyset$ and $A' \cup B = X$, so (A', B) is a separation of X. As X is connected and $A' \supseteq U' \neq \emptyset$ we see that $B = \emptyset$. Thus, every separation of Y is trivial and Y is connected.

(40) Suppose that X is locally compact Hausdorff and second countable, with a countable basis β say. Write

$$\beta' = \{U \in \beta \mid \overline{U} \text{ is compact }\}$$

This is a countable collection of precompact open sets; I claim it covers X. Indeed, suppose $x \in X$. Then as X is locally compact, there is a neighbourhood W of x such that $\overline{W}$ is compact. As β is a basis, there is a set $U \in \beta$ with $x \in U \subseteq W$. As $\overline{U}$ is closed in $\overline{W}$, it is compact, so $U \in \beta'$. Thus, for any x there is a set $U \in \beta'$ with $x \in U$ as claimed.

Now enumerate β' as $\beta' = \{V_n \mid n > 0\}$. We shall define recursively precompact open sets U_n such that

$$V_n \subseteq U_n \subseteq \overline{U_n} \subseteq U_{n+1}$$

Indeed, we can take $U_0 = \emptyset$. Suppose we have defined sets $U_0, \dots U_n$ satisfying the requirements. Then $\overline{U_n}$ is compact and covered by β' (as the whole space is) so

$$\overline{U_n} \subseteq V_{k_1} \cup \dots V_{k_m}$$

say. We take

$$U_{n+1} = V_{n+1} \cup V_{k_1} \cup \dots V_{k_m}$$

and observe that this is precompact because each V_k is.

This procedure gives us U_n for all n. As $V_n \subseteq U_n$ and the V_n cover X, we see that $\bigcup_n U_n = X$ as required. □

We next quote the theorem that X is paracompact and hence normal.

We define the narrow and wide bands

$$Y_n = \overline{U_n} \setminus U_{n-1}$$
$$Z_n = U_{n+1} \setminus \overline{U_{n-2}}$$

Note that $Z_n \cap Z_m = \emptyset$ if $|n - m| \geq 3$. We also choose (using normality) open sets Z'_n with $Y_n \subseteq Z'_n \subseteq \overline{Z'_n} \subseteq Z_n$.

As X is normal, Urysohn's lemma applies. This gives us a function

$$\phi_n \colon X \to [0, n]$$

with $\phi_n = 0$ on ${Z'_n}^c$ and $\phi_n = n$ on Y_n. This means that $\mathrm{supp}(\phi_n) \subseteq \overline{Z'_n} \subseteq Z_n$, and in particular that the family of supports is locally finite. This means that

$$\phi = \sum_n \phi_n$$

is continuous. I claim that it is also proper. Indeed, suppose that $K \subseteq \mathbb{R}$ is compact, so $K \subseteq [-n, n]$ say. As $\phi \geq \phi_m = m$ on Y_m, we see that $\phi^{-1}(K) \subseteq Y_1 \cup \ldots Y_n$ which is compact. Moreover, $\phi^{-1}(K)$ is closed by continuity. As a closed subset of a compact set, it is itself compact.

(41) Contemplate the construction which assigns to a set $C \subseteq C(X)$ the set

$$C' = C \cup \{f + g \mid f, g \in C\} \cup \{fg \mid f, g \in C\}$$

Note in particular that C' is countable if C is.

Suppose $B \subseteq C(X)$ is countable. Define recursively

$$C_0 = \mathbb{Q} \cup B$$
$$C_{n+1} = C'_n \supseteq C_n$$
$$A = \bigcup_{n=0}^{\infty} C_n$$

I claim that A is a $\mathbb{Q}$-algebra. Indeed, $\mathbb{Q} \subseteq C_0 \subseteq A$. Moreover, if $f, g \in A$ then $f, g \in C_n$ for some n and so $f + g, fg \in C_{n+1} \subseteq A$. Also, each C_n is countable (by induction) so A is countable. Thus A is a countable $\mathbb{Q}$-algebra containing B, as required.

Now let X be a compact metric space which has a countable dense subset Y. Write $d_y(x) = d(y, x)$, so $d_y \in C(X)$. Write

$$B = \{d_y \mid y \in Y\}$$

(so B is a countable subset of $C(X)$).

I claim that $B \subseteq C(X)$ is separating. Indeed, suppose $u, v \in X$ and $u \neq v$, so $\epsilon = d(u, v)/2 > 0$. As Y is dense, there is a point $y \in Y \cap B(u, \epsilon)$. Then

$$d_y(u) = d(y, u) < \epsilon$$
$$d_y(v) = d(v, y) \geq d(v, u) - d(u, y) = 2\epsilon - d(u, y) > \epsilon$$

so $d_y(u) \neq d_y(v)$ as required.

Let A be a countable $\mathbb{Q}$-algebra containing B. Then $\overline{A}$ is a ring (see the proof that the closure of a $\mathbb{R}$-algebra is a $\mathbb{R}$-algebra) and contains $\overline{\mathbb{Q}} = \mathbb{R}$. Thus $\overline{A}$ is a closed separating $\mathbb{R}$-algebra. By Stone-Weierstrass, it is all of $C(X)$. Thus A is a countable dense subset of $C(X)$.

A popular error is to suppose that X need not be a metric space. One chooses a countable dense subset Y, uses Urysohn's lemma to choose a countable set B of functions separating any pair of points in Y and then argues as above. However, B need not separate the points of X. For example, take $X = [0, 1]$ and $B = \{f \in C[0, 1] \mid f(0) = f(1)\}$. Then B separates the points of the dense subset $(0, 1)$, but does not separate 0 from 1. This shows that we need to use functions of the special form indicated above. The result is false for non-metric spaces, the simplest example being $X = \beta\mathbb{N}$, the Stone-Čech compactification of the discrete space $\mathbb{N}$. You can read about this in the book if you are interested.

(42) We need to prove that the relation $\sim$ is reflexive, symmetric and transitive. The first two are immediate. For transitivity, suppose that $x \sim y$ and $y \sim z$. Suppose that $X = A \cup B$ is a separation into disjoint open sets. By assumption either $x, y \in A$ or $x, y \in B$, and also either $y, z \in A$ or $y, z \in B$. On the other hand, A and B are disjoint so it cannot happen that $y \in A$ and $y \in B$; thus the only possibilities are $x, y, z \in A$ or $x, y, z \in B$. In either case, x and z lie in the same half of the partition. Thus $x \sim z$ as required.

The quasicomponent C containing x is the set of points y such that for every open and closed set A containing x, we also have $y \in A$. In other words,

$$C = \bigcap \{A \mid x \in A \text{ and } A \text{ is open and closed}\}$$

This is the intersection of a family of closed sets, hence is closed.

Now write $x \approx y$ if there is a connected set containing x and y, so the $\approx$-equivalence classes are by definition the components. Suppose that $x \sim y$, say $x, y \in Z$ with $Z \subseteq X$ connected. Consider a separation $X = A \cup B$ as before. Then the separation $Z = (Z \cap A) \cup (Z \cap B)$ is trivial, so wlog $Z \cap B = \emptyset$ and so $Z \subseteq A$. Thus $x, y \in A$. As this happens for every separation $X = A \cup B$, we see that $x \sim y$. It follows that the component $D = \{y \mid y \approx x\}$ containing x is a subset of the quasicomponent $C = \{y \mid y \sim x\}$. Thus every component is contained in a quasicomponent, as claimed.

I know of no natural examples in which the components and quasicomponents are different, but I can show you a contrived example if you insist. You can show that if the components are open then they are the same as the quasicomponents, and that this in turn holds whenever the space is locally connected or has only finitely many components.

(43)

(44) (a) The function $f(x) = x/(1 - x^2)$ is an order preserving bijection between the interval $(-1, 1)$ and the real line (see p. 105). By defining $f(-1) = -\infty$ and $f(1) = \infty$ we obtain an order preserving bijection between $[-1, 1]$ and X. Both spaces have the order topology, so this is a homeomorphism. Thus X is compact and Hausdorff.

(b) Suppose that $F \subseteq \mathbb{R}$ is closed. Note that the subspace topology on $\mathbb{R}$ as a subspace of X is the usual topology. Write $\overline{F}$ for the closure of F in X. Then the closure of F in $\mathbb{R}$ (which is just F) is $\overline{F} \cap \mathbb{R}$. This shows that $\overline{F} \subseteq F \cup \{\pm\infty\}$, and hence that $F \cup \{\pm\infty\} = \overline{F} \cup \{\pm\infty\}$ is closed. As a closed subspace of a compact Hausdorff space, it is thus compact.

(c) Suppose that $p(x) = \sum_{k=0}^{n} a_k x^k$ is a polynomial function. If p is constant then the question is trivial, so we may assume that $n > 0$ and that $a_n \neq 0$. For definiteness, suppose that n is odd and $a_n > 0$; the other three cases are treated similarly. We define $\overline{p}(-\infty) = -\infty$ and $\overline{p}(+\infty) = +\infty$, and of course $\overline{p}(x) = p(x)$ for finite x. Because p is continuous and $p(x) \to \infty$ as $x \to \infty$, we see that for every $K > 0$ the preimage $U = p^{-1}(K, \infty)$ is open in $\mathbb{R}$ and contains some set (L, ∞). This implies that

$$\overline{p}^{-1}((K, \infty]) = U \cup (L, \infty]$$

which is open in X. Similarly for $\overline{p}^{-1}([-\infty, K))$. As the sets $[K, \infty)$ and $[-\infty, K)$ form a subbasis for the order topology, this means that $\overline{p}$ is continuous.

(d) Suppose $F \subseteq \mathbb{R}$ is closed. Write $G = F \cup \{\pm\infty\}$, which we have shown is compact. It follows that $\overline{p}(G)$ is compact, and thus that $\overline{p}(G) \cap \mathbb{R}$ is closed in $\mathbb{R}$. On the other hand, it is easy to see that $\overline{p}(G) \cap \mathbb{R} = p(F)$.

(45) (a) It is trivial that $\emptyset$ and $\mathbb{R}^\infty$ are closed. Suppose that F_i is closed for all $i \in I$, and that $F = \bigcap_{i \in I} F_i$. For each n, we know that the intersections $F_i \cap \mathbb{R}_n$ are closed in the usual topology on $\mathbb{R}^n$. The same is thus true of the intersection of this family of sets, which is just $F \cap \mathbb{R}^n$. As this holds for all n, we see that F is closed in $\mathbb{R}^\infty$ as required. A similar argument shows that finite unions of closed sets are closed.

(b) The closed sets in the subspace topology are the sets $G = F \cap \mathbb{R}^n$ where F is closed in $\mathbb{R}^\infty$. By definition of the topology on $\mathbb{R}^\infty$, these sets G are closed in the usual topology on $\mathbb{R}^n$. Conversely, if a set $F \subset \mathbb{R}^n$ is closed in the usual topology then it is easy to see that it is also closed in $\mathbb{R}^\infty$ and hence in the subspace topology on $\mathbb{R}^n$.

(c) Suppose that $x, y \in \mathbb{R}^\infty$ and $x \neq y$. For some k we have $x_k \neq y_k$, say $x_k < a < y_k$. The sets $U = \{z \in \mathbb{R}^\infty \mid z_k < a\}$ and $V = \{z \in \mathbb{R}^\infty \mid z_k > a\}$ are open and disjoint, and $x \in U$ and $y \in V$. Thus $\mathbb{R}^\infty$ is Hausdorff.

(d) Suppose $F \subset \mathbb{R}^\infty$ is such that $F \cap \mathbb{R}^n$ is finite for each n. It is clear from the definitions that F is closed in $\mathbb{R}^\infty$. The same applies by the same argument to any subset $G \subseteq F$. Thus, every subset of F is closed in F, so also every subset is open in F. That is, F is discrete.

(e) Suppose $X \subset \mathbb{R}^\infty$ is compact. Consider a subset F of X as described. By construction, $F \cap \mathbb{R}^n$ contains at most n points. Thus, by the previous part, F is discrete. It is also a closed subset of the compact set X, and thus compact. The one point sets $\{x\}$ for $x \in F$ thus form an open cover of F, so compactness implies that F is finite. Looking back at the definition of F, this implies that $X \subset \mathbb{R}^n$ for some n.

(46) Let X be a space and $\sim$ an equivalence realtion on X. Consider the map $q\colon X \to X/\sim$ defined by $q(x) = [x]$. The quotient topology on $X/\sim$ is defined by specifying that $U \subseteq X/\sim$ is open iff $q^{-1}(U)$ is open in X.

A function $f\colon X/\sim \to Y$ is continuous iff the composite $f \circ q$ is continuous.

The one-point compactification of a space X is the set $X \sqcup \{\infty\}$, in which a set U is declared to be open iff

(a) $U \cap X$ is open in X.
(b) If $\infty \in U$ then $X \setminus U$ is compact.

Consider f and X as in the given problem. First, note that $f(f(z)) = z$ for all z, so f has inverse f and is a bijection.

We need only check continuity of f at 0 and ∞ — we are allowed to assume it elsewhere. The basic neighbourhoods of $f(0) = \infty$ are the sets

$$V_R = \{z \in \mathbb{C} \mid |z| > R\} \cup \{\infty\}$$

The preimage is

$$f^{-1}(V_R) = \{w \in \mathbb{C} \mid |w| < R^{-1}\}$$

which is certainly a neighbourhood of 0. Similarly, the preimage of a basic neighbourhood $\{z \mid |z| < \epsilon\}$ of 0 is the neighbourhood $V_{1/\epsilon}$ of ∞. It follows that f is continuous, and thus that $f^{-1} = f$ is continuous, and thus that f is a homeomorphism.

Now consider the space

$$X = (\Delta \sqcup \Delta)/\sim$$

where

$$\iota_0(z) \sim \iota_1(w) \quad \Leftrightarrow \quad zw = 1$$

Define continuous maps $g_0, g_1\colon \Delta \to \mathbb{C}_\infty$ by

$$g_0(z) = z \qquad\qquad g_1(z) = f(z) = 1/z$$

Note that

$$\text{image}(g_0) = \{z \mid |z| \leq 1\}$$
$$\text{image}(g_1) = \{z \mid |z| \geq 1\} \cup \{\infty\}$$

and $g_0(z) = g_1(w)$ iff $zw = 1$ iff $i_0(z) \sim i_1(w)$.

These maps combine to give a map

$$g\colon \Delta \sqcup \Delta \to \mathbb{C}_\infty \qquad\qquad g \circ i_0 = g_0 \qquad g \circ i_1 = g_1$$

It is clear that $a \sim b \Leftrightarrow g(a) = g(b)$ so g induces a continuous injective map

$$\tilde{g}\colon X = (\Delta \sqcup \Delta)/\sim \to \mathbb{C}_\infty$$

Moreover,

$$\text{image}(g) = \text{image}(g_0) \cup \text{image}(g_1) = \mathbb{C}_\infty$$

Thus g is surjective, hence a continuous bijection of compact Hausdorff spaces, hence a homeomorphism.

(47) We choose recursively open sets V_k such that $\overline{V_k} \subseteq U_k$ and

$$K = \bigcup_{l \leq k} V_l \cup \bigcup_{l > k} U_l$$

Having chosen V_l for $l < k$ we set

$$U'_k = \bigcup_{l < k} V_l \cup \bigcup_{l > k} U_l$$

so by the recursion hypothesis $K = U'_k \cup U_k$. In other words, $(U'_k)^c \subseteq U_k$. As $(U'_k)^c$ is closed and U_k is open and compact Hausdorff spaces are normal, there is an open set V_k with

$$(U'_k)^c \subseteq V_k \subseteq \overline{V_k} \subseteq U_k$$

As $(U'_k)^c \subseteq V_k$, we have

$$K = U'_k \cup V_k = \bigcup_{l \leq k} V_l \cup \bigcup_{l > k} U_l$$

as required for the recursion. After n steps we have

$$K = \bigcup_k V_k$$

(48) (a) We shall say that a set $U \subseteq X$ is τ_α-open iff $U \in \tau_\alpha$. The definition $\tau = \bigcap_\alpha \tau_\alpha$ simply means that a set U is τ-open iff it is τ_α-open for every α. For example, for every α we are given that τ_α is a topology so $\emptyset$ and X are τ_α-open. This means that $\emptyset$ and X are τ-open, so that τ satisfies T0. Now suppose that $\{U_i\}_{i \in I}$ is a family of τ-open sets. We need to show that $U = \bigcup_I U_i$ is τ-open. For each α, we note that each U_i is τ_α-open so (by T1 for τ_α) U is τ_α-open. As this holds for all α, we see that U is τ-open. This shows that τ satisfies T1. The proof for T2 is similar.

(b) Take

$$\tau_0 = \{(a, \infty) \mid a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$$
$$\tau_1 = \{(-\infty, b) \mid b \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$$

These are both topologies on $\mathbb{R}$, but $\tau = \tau_0 \cup \tau_1$ is not. Indeed, the sets $U = (-1, \infty)$ and $V = (-\infty, 1)$ both lie in τ, but $U \cap V = (-1, 1)$ does not, contradicting the axiom T2.

(c) There was a misprint in this question; it should have defined σ to be the topology on X with subbase $\upsilon = \bigcup_\alpha \tau_\alpha$. By an exercise which I left to you in class, this is the intersection of the family of all topologies which contain υ. In other words, a topology ρ contains σ iff ρ contains υ; but ρ contains υ iff ρ contains τ_α for every α.

(d) The largest and smallest topologies on X are respectively the discrete and indiscrete topologies:

$$\tau_{\text{dis}} = \mathcal{P}(X) = \{\text{ all subsets of } X\}$$
$$\tau_{\text{ind}} = \{\emptyset, X\}$$

(49) In the following answers we denote the collection of subsets of $\mathbb{R}$ offered as a possible topology by the letter σ.

(a) No. $\mathbb{R} \notin \sigma$, so axiom T0 fails.

(b) Yes.

(c) No. The sets $[\epsilon, \infty)$ for $\epsilon > 0$ lie in σ, but their union does not:

$$\bigcup_{\epsilon > 0} [\epsilon, \infty) = (0, \infty) \notin \sigma$$

This contradicts axiom T1.

(d) Yes. T0 holds by definition. T1 holds essentially because

$$\bigcup_I (a_i, \infty) = (\inf_I a_i, \infty)$$

Suppose I have a family $\{U_i\}_{i\in I}$ of sets in σ. We want to know that $U = \bigcup_I U_i \in \sigma$. For some i we may have $U_i = \emptyset$; these terms can be discarded without affecting the union. For other i we may have $U_i = X$; if so then $U = X \in \sigma$. If there are no i for which $U_i = X$ then the question reduces to the equation above. Similarly, T2 holds because of the equation

$$(a, \infty) \cap (b, \infty) = (\max(a, b), \infty)$$

apart from a little fiddling with exceptional cases.

(e) No. $\emptyset \notin \sigma$, so T0 fails.

(f) Yes.

(g) Yes.

Let $Y \subseteq \mathbb{Q}$ be connected. According to my definition (and that in the book) the empty set is not connected, so suppose $y \in Y$. I claim that $Y = \{y\}$. If not, then there is some $z \in Y$ with $z \neq y$ and then $x = y + (z - y)/\sqrt{2}$ is irrational and lies strictly between y and z. This means that $(-\infty, x) \cap Y$ and $(x, \infty) \cap Y$ form a nontrivial partition of Y, contrary to the assumption. Thus $Y = \{y\}$ and $\mathbb{Q}$ is totally disconnected.

Now let X denote the space $\mathbb{Z}$ with the 2-adic metric and the resulting topology. I claim that the balls

$$B(n, 2^{-k}) = \{m \in \mathbb{Z} \mid n - m \text{ is divisible by } 2^{k+1}\}$$

are both open and closed. They are open essentially by definition. To see that they are closed, consider a closure point $m \in \overline{B(n, 2^{-k})}$. The ball $B(m, 2^{-k})$ meets $B(n, 2^{-k})$, so there is some integer l with say $m - l = 2^{k+1}u$ and $n - l = 2^{k+1}v$. This implies that $m - n = 2^{k+1}(u - v)$, so $m \in B(n, 2^{-k})$ as required.

Now suppose that n and m are distinct integers, say $n - m = 2^k l$ with l odd. The ball $B = B(n, 2^{-k})$ is then an open and closed set containing n but not m, so that no connected set can contain both n and m. It follows that $\mathbb{Z}$ is totally disconnected with this topology.

(50) Recall the basic properties of ultrafilters (proved in the notes):

Proposition 0.0.1. *Let $\mathcal{W}$ be an ultrafilter.*
UP0 If $S \in \mathcal{W}$ and $T \supseteq S$ then $T \in \mathcal{W}$.
UP1 If $S_k \in \mathcal{W}$ for each k then $S_1 \cap \ldots S_n \in \mathcal{W}$.
UP2 If $S \subseteq X$ then either $S \in \mathcal{W}$ or $S^c \in \mathcal{W}$ (but not both).
UP3 If $T \subseteq X$ and $T \cap S \neq \emptyset$ for every $S \in \mathcal{W}$ then $T \in \mathcal{W}$.
UP4 If $S_1 \cup \ldots S_n \in \mathcal{W}$ then $S_k \in \mathcal{W}$ for some k.
UP5 $X \in \mathcal{W}$

(a) Suppose $\mathcal{W} = \mathcal{W}_x$ is fixed. Then the finite set $\{x\}$ is an element of $\mathcal{W}$. Conversely, suppose

$$S = \{x_0, \ldots x_n\} = \{x_0\} \cup \ldots \{x_n\} \in \mathcal{W}$$

By UP4, $\{x_k\} \in \mathcal{W}$ for some k. Write $x = x_k$. Using UP0 we find that

$$S \in \mathcal{W}_x \Leftrightarrow x \in S \Leftrightarrow \{x\} \subseteq S \Rightarrow S \in \mathcal{W}$$

Thus $\mathcal{W}_x \subseteq \mathcal{W}$. By maximality of $\mathcal{W}_x$, we conclude that $\mathcal{W}_x = \mathcal{W}$ as required.

(b) Suppose $S_1, \ldots S_n \in f_\#(\mathcal{W})$, so that $f^{-1}(S_k) \in \mathcal{W}$. By FIP for $\mathcal{W}$, we have

$$\emptyset \neq \bigcap_k f^{-1}(S_k) = f^{-1}\left(\bigcap_k S_k\right)$$

This means that $\bigcap_k S_k \neq \emptyset$ (indeed, if $x \in f^{-1} \bigcap S_k$ then $f(x) \in \bigcap S_k$). Thus $f_\#(\mathcal{W})$ at least has FIP.

Suppose $S \subseteq Y$. By UP2 for $\mathcal{W}$, we know that either $f^{-1}(S)$ or $f^{-1}(S)^c = f^{-1}(S^c)$ is an element of $\mathcal{W}$. This means that either S or S^c is an element of $f_\#(\mathcal{W})$. By proposition 0.2 of the ultrafilter notes, we deduce that $f_\#(\mathcal{W})$ is an ultrafilter as required.

(c) Suppose $f\colon X \to Y$ is continuous. Suppose $Z \subseteq Y$. As $Z \subseteq \overline{Z}$, we have $f^{-1}(Z) \subseteq f^{-1}(\overline{Z})$. By continuity, $f^{-1}(\overline{Z})$ is closed. As $\overline{f^{-1}(Z)}$ is the smallest closed set containing $f^{-1}(Z)$, we deduce that

$$\overline{f^{-1}(Z)} \subseteq f^{-1}(\overline{Z})$$

as required.
Conversely, suppose we are told that $\overline{f^{-1}(Z)} \subseteq f^{-1}(\overline{Z})$ for each $Z \subseteq Y$. In the case when Z is closed, we find that $\overline{f^{-1}(Z)} \subseteq f^{-1}(Z)$, which means that $f^{-1}(Z)$ is closed. Thus preimages of closed sets are closed, which means that f is continuous.

(d)

$$\begin{aligned} x \in \overline{Z} &\Leftrightarrow \forall U \in \mathcal{N}_x \quad U \cap Z \neq \emptyset \\ &\Leftrightarrow \mathcal{N}_x \cup \{Z\} \text{ has FIP} \\ &\Leftrightarrow \exists \text{ an ultrafilter } \mathcal{W} \quad \mathcal{N}_x \cup \{Z\} \subseteq \mathcal{W} \\ &\Leftrightarrow \exists \mathcal{W} \quad Z \in \mathcal{W} \text{ and } \mathcal{N}_x \subseteq \mathcal{W} \\ &\Leftrightarrow \exists \mathcal{W} \quad Z \in \mathcal{W} \text{ and } \mathcal{W} \to x \end{aligned}$$

(e) Suppose that f is continuous, and that $\mathcal{W} \to x$. Then

$$f^{-1}(\mathcal{N}_{f(x)}) \subseteq \mathcal{N}_x \subseteq \mathcal{W}$$

so by definition of $f_\#$ we have

$$\mathcal{N}_{f(x)} \subseteq f_\#(\mathcal{W})$$

so

$$f_\#(\mathcal{W}) \to f(x)$$

To be a bit more explicit, suppose that $U \in \mathcal{N}_{f(x)}$. By continuity $f^{-1}(U) \in \mathcal{N}_x$. As $\mathcal{W} \to x$ we have $\mathcal{N}_x \subseteq \mathcal{W}$, so $f^{-1}(U) \in \mathcal{W}$. By the definition of $f_\#$ this means that $U \in f_\#(\mathcal{W})$. This holds for every neighbourhood U of $f(x)$ so $\mathcal{N}_{f(x)} \subseteq f_\#(\mathcal{W})$.
Conversely, suppose we know that

$$\mathcal{W} \to x \quad \Rightarrow \quad f_\#(\mathcal{W}) \to f(x)$$

Suppose $Z \subseteq Y$ is closed; we want to show that $f^{-1}(Z)$ is closed. Suppose that x is a closure point of $f^{-1}(Z)$. By question (51d) above, we know that there is an ultrafilter $\mathcal{W}$ on X such that $f^{-1}(Z) \in \mathcal{W}$ and $\mathcal{W} \to x$. By hypothesis, $f_\#(\mathcal{W}) \to f(x)$. Moreover, from the definition of $f_\#$ we see that $Z \in f_\#(\mathcal{W})$. Applying the other half of (51d), we find that $f(x) \in \overline{Z} = Z$. Thus $x \in f^{-1}(Z)$. This shows that $f^{-1}(Z)$ is closed, as required.

(52) Let me first try to demystify this topology a little. If X is a metric space then it can be shown (indeed, you can show) that the Vietoris topology is derived from the metric

$$d(K, L) = \max(\sup_{x \in K} d(x, L), \sup_{y \in L} d(y, K))$$

where

$$d(x, L) = \inf_{y \in L} d(x, y) \qquad\qquad d(y, K) = \inf_{x \in K} d(x, y)$$

(a)

$$K \in s(U) \cap s(V) \Leftrightarrow (K \subseteq U \text{ and } K \subseteq V) \Leftrightarrow K \subseteq U \cap V \Leftrightarrow K \in s(U \cap V)$$

(b) Suppose $U \subseteq V$. Then
$$K \in m(U) \Leftrightarrow \emptyset \neq K \cap U \subseteq K \cap V \Rightarrow K \in m(V)$$
so $m(U) \subseteq m(V)$ and $m(U) \cap m(V) = m(U)$.

(c) We have $K \in s(U) \cap m(V)$ iff $K \subseteq U$ and there is a point $x \in K \cap V$. If so, then $x \in K \subseteq U$ and so $x \in U \cap V$. Thus $K \in s(U) \cap m(U \cap V)$. The opposite inclusion is trivial.

(d) Suppose $K \in s(U \cup V)$, so $K \subseteq U \cup V$. Then either $K \subseteq U$, or $K \cap V \neq \emptyset$. Thus
$$s(U \cup V) \subseteq s(U) \cup m(V)$$
It follows easily that
$$s(U \cup V) = s(U) \cup (s(U \cup V) \cap m(V))$$
(indeed, if $B \subseteq A \subseteq B \cup C$ then $A = B \cup (A \cap C)$ for any sets A, B and C).

By definition, a basic open set has the form
$$s(U_1) \cap \ldots \cap s(U_t) \cap m(V_1) \cap \ldots \cap m(V_r)$$
(we allow the cases $t = 0$ or $r = 0$ in which there are no s's or m's). However
$$s(U_1) \cap \ldots s(U_t) = s(U) \qquad\qquad U = \bigcap_k U_k$$
here we interpret U as X if $r = 0$; note that $s(X) = Z$. Next,
$$s(U) \cap m(V_1) \cap \ldots m(V_r) = s(U) \cap m(V_1 \cap U) \cap \ldots m(V_r \cap U)$$
so we may assume that $V_k \subseteq U$ for each k. Finally, if $V_k \subseteq V_l$ then $m(V_k) \subseteq m(V_l)$ so we can forget about V_l without affecting the union. By doing this repeatedly, we obtain a representation in which the V's are pairwise incomparable.

We next prove that Z is compact, using Alexander's subbasis theorem. Consider a covering of Z by subbasic open sets:
$$Z = \bigcup_{i \in I} s(U_i) \cup \bigcup_{j \in J} m(V_j)$$
Write
$$K = X \setminus \bigcup_{j \in J} V_j$$
Note that $K \in Z$, but $K \notin m(V_j)$ for any $j \in J$ so we must instead have $K \in s(U_a)$ for some $a \in I$. Thus $K \subseteq U_a$. Next, consider $K' = X \setminus U_a \in Z$. Note that $K' \subseteq X \setminus K = \bigcup_J V_j$ and K' is compact so $K' \subseteq \bigcup_{J'} V_j$ for some finite $J' \subseteq J$.

Now consider an arbitary element $L \in Z$. Either $L \subseteq U$ (so $L \in s(U)$) or $L \cap K' \neq \emptyset$. In the latter case we have $L \cap \bigcup_{J'} V_j \neq \emptyset$ so $L \cap V_j \neq \emptyset$ for some $j \in J'$, so $L \in m(V_j)$. Either way, we have
$$L \in s(U) \cup \bigcup_{J'} m(V_j)$$
so
$$s(U) \cup \bigcup_{J'} m(V_j) = Z$$
This is the required finite subcover.

Finally, we prove that Z is Hausdorff. Suppose $K, L \in Z$ with $K \neq L$. Without loss of generality, there is an element $x \in K \setminus L$. Choose disjoint open sets U and V with $x \in U$ and $L \subseteq V$ — there was a lemma telling us that this is possible. Then $K \in m(U)$ and $L \in s(V)$ and $m(U) \cap s(V) = \emptyset$. Thus Z is Hausdorff.