TOPOLOGY PROBLEMS

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(1) State the Arzela-Ascoli theorem for real valued functions, defining all the terms involved.

Assuming the theorem for real valued functions, deduce the evident analog for complex valued functions.

(2) Consider the discrete space $\mathbf{2} = \{0, 1\}$ and its infinite Cartesian power $X = \mathbf{2}^\omega$. The Tychonov theorem implies that X is compact. In this exercise we shall prove this in a more elementary way. First, given $\underline{a} \in X$ we write

$$\underline{a}[n] = (a_1, \dots a_n) \in \mathbf{2}^n$$

Also, given $\underline{b} \in \mathbf{2}^n$ we write

$$U(n, \underline{b}) = \{\underline{a} \in X \mid \underline{a}[n] = \underline{b}\}$$

(a) Prove that the sets $U(n, \underline{b})$ (for all finite n and all $\underline{b} \in \mathbf{2}^n$) are both open and closed, and that they form a basis for the product topology on X.
(b) Suppose (for a contradiction) that $\mathcal{U} = \{U_i \mid i \in I\}$ is an open cover of X which has no finite subcover. Observe that at least one of $U(1, 0)$ and $U(1, 1)$ also fails to be covered by any finite subfamily of $\mathcal{U}$.
(c) By choosing a_k recursively, show that there is a point $\underline{a} \in X$ such that for each k, $U(k, \underline{a}[k])$ is not covered by any finite subfamily of $\mathcal{U}$.
(d) Deduce a contradiction.

(3) Prove that if $X \subset \mathbb{R}^n$ is connected, then its closure is also connected but its interior need not be.

(4) Let X be a compact Hausdorff space, and $A \subseteq C(X)$ a closed subalgebra. Define a relation $\sim$ on X by

$$x \sim y \Leftrightarrow \forall f \in A \quad f(x) = f(y)$$

and write $Y = X/\sim$. Prove that Y is compact Hausdorff and that $C(Y)$ is isometrically isomorphic to A.

(5) Define the terms "countable" and "dense".

Consider the space

$$C(\mathbb{N}) = \{ \text{ bounded functions } f : \mathbb{N} \to \mathbb{R}\}$$

with the metric

$$d(f, g) = \sup_n |f(n) - g(n)|$$

Prove that there is no countable dense subset of $C(\mathbb{N})$.

(6) Define the compact-open topology on the set $C(X, Y)$ of continuous functions from X to Y.

State the shrinking lemma. You may wish also to state a simplified version applying to a finite cover of a compact Hausdorff space.

Suppose that X is Hausdorff and that β is a basis for Y. Prove that

$$\sigma = \{S(K, U) \mid K \subseteq X \text{ compact }, U \in \beta\}$$

is a subbasis for the compact-open topology on $C(X, Y)$.

(7) Let X be a complete metric space and $Y \subseteq X$. We consider Y as a metric space in the obvious way, so we can construct its completion $\tilde{Y}$. Prove that $\tilde{Y} \simeq \overline{Y}$.

(8) Fix α with $0 < \alpha < 1$ and let X be the set of contraction mappings $f\colon [0,1] \to [0,1]$ of ratio α. Prove that X is compact. Prove that the function

$$F\colon X \to [0,1]$$

$$F(f) = \text{ the unique fixed point of } f$$

is continuous (this has nothing to do with the compactness).

(9) Let X be a nonempty complete metric space. A contraction mapping on X with ratio $\alpha < 1$ is a function $f\colon X \to X$ such that

$$d(f(x), f(y)) \leq \alpha d(x, y)$$

A fixed point of f is a point $x \in X$ such that $f(x) = x$. Prove (by considering the sequence of iterates $f^k(a) = f(f(\dots(a))))$ that a contraction mapping has a unique fixed point. If a is an arbitary point and b is the fixed point, prove that

$$d(a, b) \leq d(a, f(a))/(1 - \alpha)$$

(10) Let $A \subset \mathbb{R}^n$ be a compact convex set, and suppose that 0 is in the interior of A. Convexity means that whenever u and v lie in A, the line segment $[u, v] = \{tu + (1-t)v \mid 0 \leq t \leq 1\}$ joining them lies wholly in A (draw some pictures).

(a) Suppose that $0 \neq a \in A$. Prove that the line segment

$$(0, a) = \{ta \mid 0 < t < 1\}$$

lies in the interior of A.

(b) Prove that the map $\text{bdy}(A) \to S^{n-1}$ sending u to $u/\|u\|$ is a homeomorphism (where $\text{bdy}(A)$ is the boundary of A).

(11) State the Baire category theorem, in a form involving closed or nowhere-dense sets.

You may assume the following (intuitively reasonable) fact:

Suppose that $X \subseteq \mathbb{R}^2$ is connected and $x \in \text{int}(X)$. Then $X \setminus \{x\}$ is connected.

Suppose that $f\colon [a, b] \to \mathbb{R}$ is continuous and injective. Prove that $\text{int}(f([a, b])) = \emptyset$.

Deduce that there is no continuous bijection $\mathbb{R} \to \mathbb{R}^2$.

(12) Let (X, d) be a metric space. Given a closed subset $Y \subseteq X$ and a point $x \in X$, write $\overline{d}(x, Y) = \inf\{d(x, y) \mid y \in Y\}$.

(a) Prove that $\overline{d}(x, Y) = 0$ if and only if $x \in Y$.

(b) Prove that $\overline{d}(x, Y)$ is continuous as a function of x.

(c) Prove that the function

$$e(a, b) = \min(d(a, b), \overline{d}(a, Y) + \overline{d}(b, Y))$$

is a pseudometric on X.

(d) Prove that if Y and Y' are disjoint closed subsets of X then there are disjoint open sets U and U' with $Y \subseteq U$ and $Y' \subseteq U'$.

(13) A subset $Y \subset X$ is said to be dense in X if the closure of Y in X is the whole of X.

(a) Give an example of a countable dense subset of $\mathbb{R}$.

(b) Prove that two continuous functions f and g from $\mathbb{R}$ to $\mathbb{R}$ which agree on a dense set are equal.

(c) Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is continuous and that $f(x) + f(y) = f(x + y)$ for all x and y in $\mathbb{R}$. Prove that $f(x) = f(1)x$ for all x.

(14) Suppose X is a metric space and $A \subseteq C(X)$. Say that A is equilipschitz if there is a constant K such that

$$|f(x) - f(y)| \leq K d(x, y)$$

for all $x, y \in X$ and $f \in A$. Prove that this implies that A is equicontinuous.

(15) Let X be a space, A a subset of $C(X)$. For any open set V we can ask whether the set $A|_V$ is equicontinuous. Show that there is a largest open set U for which this is true.

Consider the case $X = \mathbb{R}$ and $A = \{f_n \mid n \geq 2\}$ where $f_n(x) = x^n$. Show that $U = (-1, 1)$. You may assume (or prove, if you're feeling enthusiastic) that for $0 < r < 1$ and $n \geq 2$ we have

$$nr^{n-1} \leq 2/(r^{-1} - 1)$$

(16) Find examples of the following situations:
   (a) A set $X \subset \mathbb{R}$ which is equal to its boundary.
   (b) A set $X \subset \mathbb{R}$ which is not the closure of its interior.
   (c) A set $X \subset \mathbb{R}$ which is the interior of its closure.
   (d) A set $X \subset \mathbb{Q}$ which is both open and closed in $\mathbb{Q}$.
   (e) An infinite, bounded, closed set $X \subset \mathbb{R}$ with empty interior.
   (f) Subsets $X, Y \subset \mathbb{R}$ with $\overline{X \cap Y} \neq \overline{X} \cap \overline{Y}$
   (g) A sequence of open sets $U_n \subset \mathbb{R}$ for $n \in \mathbb{N}$ whose intersection is not open.

Define the terms "compact" and "Hausdorff".

Let X be a compact Hausdorff space. Consider the set

$$A = C(X, [0, 1]) = \{ \text{ continuous functions } f : X \to [0, 1]\}$$

Define the usual topology on A.

Suppose that A is compact. Prove that X is finite.

(18) For this question we use the notation

$$iA = \text{ interior of } A$$

$$kA = \text{ closure of } A$$

$$cA = \text{ complement of } A$$

It is interesting to ask what sets we can get by starting with a given set A and repeatedly applying the operators i, k, and c.
   (a) "Simplify" the following expressions:

$$ccA \quad kkA \quad iiA \quad cicA \quad ckcA$$

   (b) Prove that if A is the closure of some open set U, then $A = kiA$.
   (c) Prove that $kikiA = kiA$ for any A, and hence that $ikikB = ikB$ for any B.
   (d) Prove that for any set A, at most fourteen different sets (including A itself) can be obtained from A by repeatedly applying the operations i,k, and c. Seven of these are "roughly the same as A" and the other seven "roughly the same as cA".
   (e) Find a subset $A \subset \mathbb{R}$ such that all fourteen of these sets are different. Hints: If you take care of the first seven, the other seven will probably take care of themselves. You will want to build A out of several different chunks spaced out along the real line.

(19) Now write

$$bA = \text{ boundary of } A = kA \cap kcA$$

   (a) "Simplify" the following expressions:

$$k(A \cup B) \quad i(A \cap B) \quad c(A \cap B) \quad A \cap (B \cup C) \quad bcA \quad kbA$$

   (b) Prove that if A is closed, then $ibA = \emptyset$, and thus that $b^2 A = bA$.
   (c) Prove that $b^3 A = b^2 A$ for any set A.
   (d) Find a set $A \subset \mathbb{R}$ with $A \neq bA \neq b^2 A$.
   (e) What does it mean to have $bA = \emptyset$?

(20) Prove that if X is Hausdorff then every finite subset (in particular, every set consisting of a single point) is closed.

Let Y be the set $C[0, 1]$ of continuous functions $f : [0, 1] \to \mathbb{R}$ from the closed unit interval to $\mathbb{R}$. We regard Y itself as a topological space, with the topology coming from the following metric:

$$d(f, g) = \int_0^1 |f(x) - g(x)| dx$$

(you may assume that this is a metric and not merely a pseudometric). Prove that Y is Hausdorff.

Define an equivalence relation $\sim$ on Y by

$$f \sim g \Leftrightarrow \exists \epsilon > 0 \; \forall x \in [0, \epsilon] \quad f(x) = g(x)$$

In other words, f and g are equivalent iff they agree on some interval $[0, \epsilon]$ with $\epsilon > 0$.

We write 1 for the constant function with value 1 and so on. Show that for any $\delta > 0$ there is a function $f \in Y$ such that $f \sim 1$ but $d(f, 0) < \delta$.

Conclude that in the quotient space $X = Y/\sim$ (with the quotient topology, of course) 0 and 1 cannot be separated by disjoint open sets. This means that X is not Hausdorff. It is actually indiscrete, but you need not prove this.

(21) The graph of a function $f : X \to Y$ is the set

$$\Gamma(f) = \{(x, f(x)) \mid x \in X\} \subseteq X \times Y$$

This is just the "rule" of f, in the terminology of section 1.2. Prove that if Y is Hausdorff and f is continuous then $\Gamma(f)$ is closed. Try to make it a slick proof. Find an example of a discontinuous function $f : \mathbb{R} \to \mathbb{R}$ with closed graph.

(22) Define carefully the terms "quotient topology" and "one-point compactification".

Consider the spaces

$$S^3 = \{(z, w) \in \mathbb{C}^2 \mid \quad |z|^2 + |w|^2 = 1\}$$

$$S^2 = \mathbb{C}_\infty$$

Define maps

$$f : S^2 \to S^2 \qquad f(z) = 1/z$$

$$g : S^3 \to S^2 \qquad g(z, w) = z/w$$

(with the usual conventions about 0 and ∞). Prove that f is a homeomorphism and that g is continuous. You may assume that the following maps are continuous:

$$h = f|_{\mathbb{C}\setminus\{0\}} : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\} \qquad h(z) = 1/z$$

$$m : \mathbb{C} \times \mathbb{C} \to \mathbb{C} \qquad m(z, w) = zw$$

(23) This exercise has very little to do with topology, but it may help you understand ultrafilters. Let $\mathcal{W}$ be a free ultrafilter on $\mathbb{N}$. Define an equivalence relation on the product space $\mathbb{R}^{\mathbb{N}}$ by

$$a \sim b \text{ iff } \{n \in \mathbb{N} \mid a_n = b_n\} \in \mathcal{W}$$

The quotient set $\mathbb{R}^* = \mathbb{R}^{\mathbb{N}}/\sim$ is called the field of hyperreal numbers. Define an order relation and algebraic operations on $\mathbb{R}^*$ as follows:

$$\begin{aligned} [a] < [b] &\Leftrightarrow \{n \mid a_n < b_n\} \in \mathcal{W} \\ [a] + [b] &= [a + b] \\ [a][b] &= [ab] \end{aligned}$$

Here ab means the sequence with n'th term $a_n b_n$ and so on. We shall confuse a real number x with the equivalence class of the constant sequence $(x, x, \ldots)$.

(a) Prove that $\sim$ is an equivalence relation.
(b) Prove that the order and the operations are well defined, so that if $[a] = [a']$ and $[b] = [b']$ then $[ab] = [a'b']$ etc.
(c) Check that $\mathbb{R}^*$ is an ordered field, so it has properties (1) to (8) listed on pages 30–31 of the book. You need only check a small random selection.
(d) Say that $a \in \mathbb{R}^*$ is infinitesimal if $-1/n < a < 1/n$ for every $n \in \mathbb{Z}_+$. Write I for the set of infinitesimals. Show that $[1, 1/2, 1/3, \ldots] \in \mathbb{R}^*$ is a nonzero infinitesimal.

(e) Say that $a \in \mathbb{R}^*$ is finite if $-n < a < n$ for some $n \in \mathbb{Z}_+$. Write F for the set of finite hyperreals. Write $a \simeq b$ iff $a - b \in I$. Show that the composite

$$\mathbb{R} \to F \to F/\simeq$$

is bijective.

(24) Suppose that $k\colon [0,1] \times [0,1] \to \mathbb{R}$ is continuous. For fixed x we can consider $k(x,y)$ as a function of y. More formally, define

$$k^{\#}(x)(y) = k(x,y)$$

so

$$k^{\#}(x)\colon [0,1] \to \mathbb{R}$$

Prove that $k^{\#}(x)$ is continuous, so that $k^{\#}(x) \in C(X)$ and thus

$$k^{\#}\colon [0,1] \to C[0,1]$$

Prove that this map $k^{\#}$ is continuous.

Now define

$$K\colon C[0,1] \to C[0,1]$$

$$(Ku)(x) = \int_0^1 k(x,y)u(y)\mathrm{d}y$$

Prove that Ku is indeed continuous as is implicit in the notation. Prove that K is continuous.

(25) Define the term "equicontinuous".

Let A be the space of continuous functions $f\colon [0,1] \to \mathbb{R}$ with the usual topology.

Define a function $K\colon A \to A$ by

$$(Kf)(t) = \int_0^1 (s+t)^2 f(s)\mathrm{d}s$$

You may assume that Kf is continuous, so it lies in A as advertised. Prove that K is continuous.

Let B be the unit ball:

$$B = \{f \in X \quad | \quad \|f\| \leq 1\}$$

and V the eigenspace of eigenvalue 1:

$$V = \{f \in X \mid Kf = f\}$$

Prove that $V \cap B$ is compact.

(26) Suppose that $f\colon X \to Y$ is continuous and surjective, and that X is locally connected. Need Y be locally connected ? Give a proof or counterexample.

(27) (a) A topological manifold of dimension n is by definition a space M such that
   (i) M is Hausdorff
   (ii) M is second countable
   (iii) Every point $x \in M$ has a neighbourhood U which is homeomorphic to $\mathbb{R}^n$ (M is "locally euclidean").

   For example, the sphere, the torus and the Möbius band are all two dimensional manifolds (you need not prove this).

   Show that an open subset of $\mathbb{R}^n$ (and hence, of any manifold) is a manifold.

(b) Show that the components of a manifold are path-connected manifolds.

(c) Show that a manifold is locally compact, and hence (quoting a suitable theorem) that it is paracompact.

(d) Give an example of a space satisfying parts (a) and (b) but not (c) of the definition of a manifold, and similarly for the other two cases.

(28) Define

$$M_n = \{n \times n \text{ matrices of real numbers }\}$$

$$GL_n = \{ \text{ invertible matrices } \} \subset M_n$$

$$O_n = \{A \in GL_n \mid A^T = A^{-1}\}$$

$$SO_n = \{A \in O_n \mid \det(A) = 1\}$$

Where A^T is the transpose of the matrix A. We topologise all these spaces as subsets of $M_n \simeq \mathbb{R}^{n^2}$, and we use the Euclidean norm on $\mathbb{R}^n$:

$$\|\underline{x}\| = \left(\sum_k x_k^2\right)^{1/2}$$

The operator norm of a matrix A is defined as

$$\|A\|_{op} = \sup\{\|A\underline{x}\| \ \mid \ \|x\| \leq 1\}$$

(a) Prove that the determinant map $\det\colon M_n \to \mathbb{R}$ is continuous.
(b) Prove that GL_n is an open, disconnected subset of M_n. (A harder problem is to prove that GL_n has precisely two components).
(c) Prove that SO_2 is homeomorphic to a circle.
(d) Prove that O_3 is compact (= bounded and closed).
(e) Prove that there is a constant K (depending only on n) such that

$$\|A\|_{op} \leq K \max\{|A_{kl}| \mid 1 \leq k, l \leq n\} = K\|A\|_\infty$$

(f) Prove that $\|A\|_{op}$ is a continuous function of A.
(g) If you are prepared to use Rouché's theorem from complex analysis, then you can prove that the spectral radius $r(A)$ is a continuous function of A:

$$r(A) = \max\{|\lambda| \ \mid \ \lambda \text{ is an eigenvalue of} A\}$$

(29) Let $f\colon \mathbb{C} \to \mathbb{C}$ be a non-constant polynomial. You may assume the fact (proven in complex analysis) that f is open, surjective and (of course) continuous. Write

$$U = \{z \in \mathbb{C} \mid \quad |z| > 1\}$$

$$V = f^*(U) = \{z \in \mathbb{C} \mid \quad |f(z)| > 1\}$$

This exercise shows that V is connected.

(a) Let $D \subset \mathbb{C}$ be open. Prove that the connected components of D are open and closed in D.
(b) Prove that if $K \subset \mathbb{C}$ is bounded and closed, then the same is true of $f(K)$.
(c) Prove that there are positive real constants K, L such that

$$|f(z)| > L \Rightarrow |z| > K \Rightarrow |f(z)| > 1$$

(d) Prove that V has precisely one unbounded component.
(e) Suppose that W is bounded component of V. Prove using part (a) that $\overline{f(W)} \cap U = f(W)$, and thus that $f(W)$ is open, closed and bounded in U. Deduce a contradiction and conclude that V is connected.
(f) Prove that the complement of the Mandelbrot set is connected. This means that in some sense the Mandelbrot set has no holes, for a hole would be a bounded component of the complement. A reminder of the definition:

$$q_c(z) = z^2 + c$$

$$f_0(c) = 0$$

$$f_{n+1}(c) = q_c(f_n(c)) = f_n(c)^2 + c$$

$$M = \{c \in \mathbb{C} \mid |f_n(c)| \leq 2 \text{ for all } n\}$$

(30) Given a complex number c, define $q_c(z) = z^2 + c$ and

$$f_n(c) = q_c(q_c(q_c \dots (0) \dots))$$

where q_c is applied n times. In other words:

$$f_0(c) = 0$$

$$f_{n+1}(c) = q_c(f_n(c)) = f_n(c)^2 + c$$

(a) Give formulae for $f_1(c)$, $f_2(c)$ and $f_3(c)$.
(b) Prove that $f_n\colon \mathbb{C} \to \mathbb{C}$ is continuous. A little care is needed here to avoid talking nonsense.
(c) The Mandelbrot set M is defined as

$$M = \{c \in \mathbb{C} \mid |f_n(c)| \leq 2 \text{ for all } n\}$$

Prove that M is bounded and closed.

Later on, we shall prove some other topological properties of M, including the fact that it has no holes. It is also known that M is connected, but this is a deep result, relying on some substantial work in complex analysis. It is also known that M is the closure of its interior. One of the most important questions about M, which I believe is still unanswered, is whether it locally connected. It is known that many interesting things would follow from this.

(31) Let X be a compact Hausdorff space, and $C(X)$ the set of continuous functions $u\colon X \to \mathbb{R}$. Define two maps b and t ("bottom" and "top") from $C(X)$ to $\mathbb{R}$ by

$$b(u) = \min\{u(x) \mid x \in X\}$$

$$t(u) = \max\{u(x) \mid x \in X\}$$

(why are these finite ?). Prove that t and b are continuous.

A measure on X is defined to be a map $\mu\colon C(X) \to \mathbb{R}$ such that
(a) (linearity) $\mu(au + bv) = a\mu(u) + b\mu(v)$ (where $a, b \in \mathbb{R}$ and $u, v \in C(X)$).
(b) (positivity) if $u \geq 0$ (i.e. $u(x) \geq 0$ for all x) then $\mu(u) \geq 0$.
(c) (normalisation) $\mu(1) = 1$ (where the first 1 is the constant function).

Here are some examples with $X = [0, 1]$:
(a) $\delta(u) = u(0)$ (the Dirac measure).
(b) $\lambda(u) = \int_0^1 u(x)dx$ (Lebesgue measure).
(c) $\rho_n(u) = \frac{1}{n+1}\sum_{k=0}^{n} u(k/n)$

We shall consider the following infinite product space:

$$Z = \prod_{u \in C(X)} [b(u), t(u)] \subset \prod_{u \in C(X)} \mathbb{R}$$

(a) Describe the elements of Z. This is not supposed to be a deep question; just transcribe the general definition of an infinite product and perhaps tidy up the result a bit.
(b) Give two examples of elements of Z.
(c) Show that if μ is a measure then $b(u) \leq \mu(u) \leq t(u)$.
(d) Show that every measure μ is a continuous map $C(X) \to \mathbb{R}$. (This is not actually relevant to what follows).
(e) Define an injective function $j\colon M(X) \to Z$ (there is only one reasonable one).
(f) Given $u, v \in C(X)$ consider the set

$$A_{u,v} = \{\alpha \in Z \mid \pi_u(\alpha) + \pi_v(\alpha) = \pi_{u+v}(\alpha)\}$$

Show that $A_{u,v}$ is closed in Z and contains $j(M(X))$. By elaborating on this idea, show that $j(M(X))$ is closed in Z and thus compact. We identify $M(X)$ with $j(M(X))$ and give it the subspace topology.
(g) Show that $\mu_n \to \mu$ in $M(X)$ iff $\mu_n(u) \to \mu(u)$ for every $u \in C(X)$. In particular, you can show that $\rho_n \to \lambda$ in $M([0, 1])$.

(0) Which of the following pairs (X, d) is a metric space ?
(a) $X = \mathbb{R}^n$, $\quad d(\underline{x}, \underline{y}) = \sum_{k=1}^{n} k|x_k - y_k|$

(b) $X = \mathbb{R}$, $\quad d(x,y) = (x-y)^2$

(c) $X = \mathbb{R}$, $\quad d(x,y) = \begin{cases} \min(|x-y|,1) & \text{if } x-y \in \mathbb{Q} \\ 1 & \text{if } x-y \notin \mathbb{Q} \end{cases}$

(d) $X = \mathbb{R}^n$, $\quad d(\underline{x},\underline{y}) = \min_k |x_k - y_k|$

(e) $X = \mathbb{Q}$. If $x = y$ we take $d(x,y) = 0$, otherwise we can write $x - y$ as $2^n a/b$ where a and b are odd integers and n is also an integer. In this case we take $d(x,y) = 2^{-n}$.

(f) $X = \mathbb{Z}$, $\quad d(x,x) = 0$. If $x \neq y$ write $x - y = 3^n a$ where a is an integer not divisible by 3, and take $d(x,y) = 3^n$.

(1) Suppose that $\|x\|$ is a norm on $\mathbb{R}^n$, so that

$$\|tx\| = |t|.\|x\| \qquad (t \in \mathbb{R} \quad x \in \mathbb{R}^n)$$
$$\|x+y\| \leq \|x\| + \|y\|$$
$$\|x\| \geq 0$$
$$\|x\| = 0 \Rightarrow x = 0$$

We shall use the symbol $\|x\|_2$ for the usual Euclidean norm:

$$\|x\|_2 = \left(\sum x_k^2\right)^{1/2}$$

You may assume that the topology derived from the Euclidean metric is the same as the product toplogy.

By considering the norms of the basis vectors, prove that the function $n(x) = \|x\|$ is continuous (with the product topology on $\mathbb{R}^n$). By considering the sphere in $\mathbb{R}^n$, prove that there are constants k and K such that

$$k\|x\|_2 \leq \|x\| \leq K\|x\|_2$$

Deduce that the topology on $\mathbb{R}^n$ derived from the metric $d(x,y) = \|x-y\|$ is the usual one.

(2) Suppose that X is compact and Hausdorff, and that $K, L \subseteq X$ are disjoint closed sets. Prove that there exist disjoint open sets U and V such that $K \subseteq U$ and $L \subseteq V$.

(3) For which of the following pairs of sets $Y \subseteq X$ is Y open in X ?

(a) $X = \mathbb{R}$, $\quad Y = \mathbb{R} \setminus \mathbb{Z} = \{x \in \mathbb{R} \mid x \notin \mathbb{Z}\}$

(b) $X = [-1,1]$, $\quad Y = [0,1]$

(c) $X = \mathbb{Q}$, $\quad Y = \mathbb{Q} \cap [-\sqrt{2}, \sqrt{2}]$

(d) $X = \mathbb{R}$, $\quad Y = \{x \in \mathbb{R} \mid x \neq 1/n \text{ for any } n \in \mathbb{Z}_+\}$

(e) $X = \{1/n \mid n \in \mathbb{Z}_+\}$, $\quad Y = \{1/(n+1) \mid n \in \mathbb{Z}_+\}$

(f) $X = [0,1] \cup [2,3]$, $\quad Y = [0,1]$

(4) A function $f\colon X \to Y$ is said to be open if $f(U)$ is open for every open set U. You may recall from complex analysis that non-constant analytic functions are open. Closed functions are defined similarly. In the following questions, give a proof or a counterexample:

(a) Must a closed function be continuous ?

(b) Must a continuous function be open ?

(c) Must a homeomorphism be closed ?

(d) Must a continuous function be closed ?

(e) Must a continuous function of compact Hausdorff spaces be closed ?

(5) Given a compact Hausdorff space X, let $C(X)$ denote the set of continuous functions $u\colon X \to \mathbb{R}$. We define a metric on $C(X)$ by

$$d(u,v) = \sup\{|u(x) - v(x)| \mid x \in X\}$$

(you should at least work out in your head why this is a metric). Suppose that Y is another compact Hausdorff space, and that $f\colon X \to Y$ is continuous. Show that the map

$$f^*\colon C(Y) \to C(X) \qquad f^*(u) = u \circ f$$

is continuous.

Now suppose X is a metric space (we shall use the same symbol d for all metrics). Define the ϵ-oscillation of u as

$$\mathrm{osc}_\epsilon(u) = \sup\{|u(x) - u(y)| \mid d(x, y) < \epsilon\}$$

Give a clean proof that $\mathrm{osc}_\epsilon\colon C(X) \to \mathbb{R}$ is continuous.

This shows that the set

$$U(\epsilon, \delta) = \{u \mid \mathrm{osc}_\epsilon(u) < \delta\}$$

is open. Prove that for fixed δ, we have

$$\bigcup_{\epsilon>0} U(\epsilon, \delta) = C(X)$$

These are the first steps in the proof of the following uniform Fourier approximation theorem. Let P be the space of functions $u\colon [-\pi, \pi] \to \mathbb{R}$ given by a finite Fourier series:

$$u(x) = \sum_{k=0}^{n} a_k \cos(kx) + b_k \sin(kx)$$

Then, given $\delta > 0$ there is a continuous map

$$F\colon C[-\pi, \pi] \to P$$

such that $d(u, F(u)) < \delta$ for all u. Of course, F is something like the Fourier transform, but we have to work out how to fix it up so that we take different but finite numbers of terms for different functions u, and have the result depending continuously on u.

(6) (a) Let p be a prime number, and consider the space $\mathbb{Z}$ with the p-adic metric:

$$v(n) = \max\{k \mid n \text{ is divisible by } p^k\}$$
$$|n| = p^{-v(n)}$$
$$d(n, m) = |n - m|$$

The exceptional cases are: $v(0) = \infty$, $|0| = d(n, n) = 0$. Let $\mathbb{Z}_p$ be the completion of $\mathbb{Z}$ with this metric.

Prove that $\mathbb{Z}_p$ is compact and Hausdorff.

(b) For each $k \in \mathbb{N}$, define a relation $\sim_k$ on $\mathbb{Z}$ by

$$n \sim_k m \text{ iff } v(n - m) \geq k$$

Prove that this is an equivalence relation. Write $\mathbb{Z}/p^k$ for $\mathbb{Z}/\sim_k$ and $[n]_k$ for the $\sim_k$-equivalence class of n. Prove that $\mathbb{Z}/p^k$ is finite and discrete in the quotient topology.

(c) Prove that the map

$$r_k\colon \mathbb{Z}/p^k \to \mathbb{Z}/p^{k-1} \qquad r_k([n]_k) = [n]_{k-1}$$

is well-defined (explain what might be a problem and why it isn't).

(d) Consider the space

$$X = \{a \mid \forall k > 0 \quad r_k(a_k) = a_{k-1}\} \subset \prod_{k\in\mathbb{N}} \mathbb{Z}/p^k$$

Prove that X is compact and Hausdorff. It is possible to prove compactness directly, but I recommend the use of general theorems instead.

(e) For each $k \in \mathbb{N}$ and $c \in \mathbb{Z}/p^k$ consider the set

$$U_k(c) = \pi_k^{-1}\{c\} \cap X = \{a \in X \mid a_k = c\}$$

Show that the sets $U_k(c)$ form a basis for the topology on X.

(f) For $a, b \in X$ define

$$v(a) = \max\{k \mid a_k = [0]_k\}$$
$$|a| = p^{-v(a)}$$
$$d(a, b) = |a - b|$$

with the obvious conventions in exceptional cases. You should work out why this is a metric, although you need not write it down. Observe in particular that $l \leq v(a) \Rightarrow a_l = [0_l]$.

Prove that the metric topology is the same as the topology considered previously.

(g) Construct an isometric embedding $f : \mathbb{Z} \to X$, and hence a homeomorphism $\mathbb{Z}_p \to X$. Prove that your map is a homeomorphism.

(7) Define what it means for a topological space to be connected.

Given $a \in \mathbb{R}^2$ and $\epsilon > 0$ prove that

$$U = \{b \in \mathbb{R}^2 \mid 0 < d(a, b) < \epsilon\}$$

is connected. You may assume the continuity of the usual functions considered in analysis.

Suppose that $X \subseteq \mathbb{R}^2$ is connected and $a \in \mathrm{int}(X)$. Prove that $X \setminus \{a\}$ is connected.

(8) Suppose that X is locally compact Hausdorff and second countable. Prove that there is a sequence of open sets U_n such that $\overline{U_n}$ is compact and contained in U_{n+1}, and $\bigcup_n U_n = X$.

For the rest of the question you may (as always) quote results proved in lectures, including those which rely on the result you have just proved. You may also wish to recall some details of the proofs of such results.

A map $f : X \to Y$ is said to be proper iff it is continuous and the preimage of every compact set is compact. Prove that there is a proper map $f : X \to \mathbb{R}$.

(9) Let X be a space, and A a subset of $C(X)$. We identify a real number with the corresponding constant function, so $\mathbb{R} \subseteq C(X)$. We shall say that A is

(a) a ring if $\mathbb{Z} \subseteq A$ and

$$f, g \in A \;\Rightarrow\; f + g \in A \text{ and } fg \in A$$

(b) a $\mathbb{Q}$-algebra if it is a ring and $\mathbb{Q} \subseteq A$

(c) an $\mathbb{R}$-algebra if it is a ring and $\mathbb{R} \subseteq A$

Contemplate the construction which assigns to a set $C \subseteq C(X)$ the set

$$C' = C \cup \{f + g \mid f, g \in C\} \cup \{fg \mid f, g \in C\}$$

Suppose $B \subseteq C(X)$ is countable. Prove that there is a countable $\mathbb{Q}$-algebra A such that $B \subseteq A \subseteq C(X)$.

Let X be a compact metric space which has a countable dense subset. Prove that $C(X)$ has a countable dense subset.

(10) Given a space X, say $x \sim y$ iff there is no separation $X = A \cup B$ into disjoint open sets such that $x \in A$ and $y \in B$. Prove that this is an equivalence relation. We shall call the equivalence classes quasicomponents. Show that each quasicomponent is closed. Show that each component is contained in a quasicomponent.

(11) Suppose that R is an equivalence relation on a space X. Show that X/R is Hausdorff iff any two unrelated points x, y with $x \not R y$ have saturated open neighbourhoods U and V with $U \cap V = \emptyset$. Note that R is officially a subset $R = \{(x, y) \mid xRy\}$ of X^2. Show that if X/R is Hausdorff then R is closed in X^2. Finally, if X is compact and R is closed in X^2, prove that X/R is Hausdorff — you will need ultrafilters for this.

(12) Consider the set $X = \mathbb{R} \cup \{\pm\infty\}$, ordered in the obvious way, and given the order topology.

(a) Prove that X is compact and Hausdorff (possibly by showing that it is homeomorphic to something else).

(b) More generally, show that if $F \subseteq \mathbb{R}$ is closed then $F \cup \{\pm\infty\}$ is a compact subspace of X.

(c) Prove that any polynomial function $p\colon \mathbb{R} \to \mathbb{R}$ extends uniquely to a continuous function $\bar{p}\colon X \to X$.
(d) Conclude that $p(F)$ is closed for every closed set $F \subset \mathbb{R}$.

(13) Let $\mathbb{R}^\infty$ ($\neq \mathbb{R}^\omega$) denote the set of sequences $\underline{a} = (a_n)_{n=0}^\infty$ such that $a_n \in \mathbb{R}$ and $a_n = 0$ for all but a finite number of indices n. (Thus $(1, 2, 3, 4, 0, 0, 0 \ldots) \in \mathbb{R}^\infty$ but $(2^{-n}) \notin \mathbb{R}^\infty$). We also write

$$\mathbb{R}^n = \{\underline{a} \in \mathbb{R}^\infty \mid a_k = 0 \text{ for } k \geq n\}$$

so that

$$\mathbb{R}^\infty = \bigcup_{n \geq 0} \mathbb{R}^n$$

We want to describe a topology on $\mathbb{R}^\infty$; we can do this by specifying the closed sets. We shall say that a subset $F \subseteq \mathbb{R}^\infty$ is closed iff $F \cap \mathbb{R}^n$ is closed in the usual topology of $\mathbb{R}^n$ for every n. You will find this exercise easier if you work with closed sets rather than open ones throughout. The space $\mathbb{R}^\infty$ is useful in algebraic topology.
(a) Prove that this gives a topology on $\mathbb{R}^\infty$.
(b) Prove that the topology which $\mathbb{R}^n$ inherits as a subspace of $\mathbb{R}^\infty$ is its usual topology.
(c) Prove that $\mathbb{R}^\infty$ is Hausdorff.
(d) Suppose $F \subset \mathbb{R}^\infty$ is such that $F \cap \mathbb{R}^n$ is finite for every n. Prove that F is discrete, and closed in $\mathbb{R}^\infty$.
(e) Suppose that $X \subseteq \mathbb{R}^\infty$ is compact. From each set $X \cap (\mathbb{R}^n \setminus \mathbb{R}^{n-1})$ which happens to be nonempty, choose an element x_n. By considering the set F of the x_n's, show that $X \subseteq \mathbb{R}^m$ for some m.

(14) Let X be a space and $\sim$ an equivalence relation on X. We give the set $X/\sim$ the quotient topology. Define this topology. State a criterion for a map $f\colon X/\sim \to Y$ to be continuous.
Define the one-point compactification X_∞ of a space X (together with its topology).
Consider the map

$$f\colon \mathbb{C}_\infty \to \mathbb{C}_\infty$$
$$f(z) = 1/z$$

(with the obvious conventions about 0 and ∞). Prove that f is a homeomorphism. You may assume that the restriction of f to $\mathbb{C} \setminus \{0\}$ is continuous.
Consider the spaces

$$\Delta = \{z \in \mathbb{C} \mid |z| \leq 1\}$$
$$X = (\Delta \sqcup \Delta)/\sim$$

where

$$\iota_0(z) \sim \iota_1(w) \quad \Leftrightarrow \quad zw = 1$$

Prove that X is homeomorphic to $\mathbb{C}_\infty$

(15) What does it mean for a space to be normal ? Prove that a compact Hausdorff space is normal.
Suppose K is compact Hausdorff, and that $K = U_1 \cup \ldots U_n$ is a finite open cover. Prove that there are open sets V_k such that $K_k = \overline{V_k} \subseteq U_k$ and $K = \bigcup_k K_k$.

(16) Let $\{\tau_\alpha\}$ be a collection of topologies on a fixed set X.
(a) Prove that $\tau = \bigcap_\alpha \tau_\alpha$ is a topology on X.
(b) Give an example of two topologies τ_0 and τ_1 on $\mathbb{R}$ such that $\tau_0 \cup \tau_1$ is not a topology.
(c) Let σ be the topology on X with subbase $\bigcap_\alpha \tau_\alpha$. Show that a topology ρ on X satisfies $\rho \supseteq \tau_\alpha$ for all α if and only if it satisfies $\rho \supseteq \sigma$. In other words, σ is the coarsest topology which is finer than each τ_α.
(d) What are the largest and smallest topologies on X ?

(17) Let τ denote the usual topology on $\mathbb{R}$. Which of the following collections of subsets of $\mathbb{R}$ form topologies ?
(a) $\{U \subseteq \mathbb{R} \mid U \subseteq \mathbb{Q}\}$

(b) $\{U \subseteq \mathbb{R} \mid U \cap \mathbb{Q} = V \cap \mathbb{Q}$ for some $V \in \tau\}$
(c) $\{[a, \infty) \mid a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$
(d) $\{(a, \infty) \mid a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$
(e) $\{U \subseteq \mathbb{R} \mid 1 \in U\}$
(f) $\{U \subseteq \mathbb{R} \mid 0 \notin U$ or $1 \in U\}$
(g) $\{U \subseteq \mathbb{R} \mid x \in U \iff x + 1 \in U\}$

(18) A space X is said to be totally disconnected iff the only connected subsets are single points. Prove that this holds for $X = \mathbb{Q}$ with its usual topology as a subspace of $\mathbb{R}$.

Recall the definition of the 2-adic metric on $\mathbb{Z}$. Any nonzero integer n can be written in a unique way as $2^k l$ where l is odd. We then define $|n|_2 = 2^{-k}$. We also let $|0|_2 = 0$, and define $d(n, m) = |n - m|_2$.

Prove that $\mathbb{Z}$ with the 2-adic topology is also totally disconnected.

(19) Here, for ease of reference, is a list of properties of ultrafilters. You may quote them without proof.

Proposition 0.0.1. *Let $\mathcal{W}$ be an ultrafilter.*
UP0 If $S \in \mathcal{W}$ and $T \supseteq S$ then $T \in \mathcal{W}$.
UP1 If $S_k \in \mathcal{W}$ for each k then $S_1 \cap \dots S_n \in \mathcal{W}$.
UP2 If $S \subseteq X$ then either $S \in \mathcal{W}$ or $S^c \in \mathcal{W}$ (but not both).
UP3 If $T \subseteq X$ and $T \cap S \neq \emptyset$ for every $S \in \mathcal{W}$ then $T \in \mathcal{W}$.
UP4 If $S_1 \cup \dots S_n \in \mathcal{W}$ then $S_k \in \mathcal{W}$ for some k.
UP5 $X \in \mathcal{W}$

(a) Show that an ultrafilter $\mathcal{W}$ is fixed (that is, equal to $\mathcal{W}_x$ for some x) iff there is a finite set $S \in \mathcal{W}$.

(b) Suppose that $\mathcal{W}$ is an ultrafilter on X and $f : X \to Y$. Define
$$f_\#(\mathcal{W}) = \{S \subseteq Y \mid f^{-1}(S) \in \mathcal{W}\}$$
Prove that this is an ultrafilter on Y. You should think of $\mathcal{W}$ as analogous to a sequence $(x_n)_{n \in \mathbb{N}}$, in X, and $f_\#(\mathcal{W})$ as analogous to the sequence $(f(x_n))_{n \in \mathbb{N}}$.

(c) Prove that f is continuous iff for each $Z \subseteq Y$ we have
$$\overline{f^{-1}(Z)} \subseteq f^{-1}(\overline{Z})$$
(The proof does not involve ultrafilters).

(d) Now suppose $Z \subseteq X$. Prove that $x \in \overline{Z}$ iff there is an ultrafilter $\mathcal{W}$ on X such that $Z \in \mathcal{W}$ and $\mathcal{W} \to x$.

(e) Prove that f is continuous iff for each ultrafilter $\mathcal{W}$ on X with $\mathcal{W} \to x$ we have $f_\#(\mathcal{W}) \to y$.

(20) Let X be a compact Hausdorff space. Let Z denote the set of all closed subsets of X (also called ζ elsewhere). We have some notion of what it means for two closed subsets of X to be approximately the same, which suggests that we might be able to define a topology on Z itself. Given an open set $U \subseteq X$, we define
$$s(U) = \{K \in Z \mid K \subseteq U\}$$
$$m(U) = \{K \in Z \mid K \cap U \neq 0\}$$
(Mnemonic: $s(U)$ stands for "subset of U", and $m(U)$ for "meets U").

Prove (or observe) the following:
$$\begin{aligned} s(U) \cap s(V) &= s(U \cap V) \\ m(U) \cap m(V) &= m(U) \quad \text{if } U \subseteq V \\ s(U) \cap m(V) &= s(U) \cap m(U \cap V) \\ s(U \cup V) &= s(U) \cup (s(U \cup V) \cap m(V)) \end{aligned}$$

Let σ' be the collection of sets of the form $s(U)$ or $t(V)$. This is trivially a subbasis; write β' for the corresponding basis and τ' for the resulting topology on Z. This is called the Vietoris topology. We shall use symbols U', V' etc for subsets of Z, to avoid confusion with subsets of X.

Show that any basic open set $U' \in \beta'$ can be written in the form

$$U' = s(U) \cap m(V_1) \cap \ldots m(V_r)$$

where U and the V_k's are open in X, the sets V_k are pairwise incomparable (i.e. for $k \neq l$ we have $V_k \not\subseteq V_l$) and $V_k \subseteq U$ for each k.

Using Alexander's subbasis theorem, show that Z is compact.

Prove that Z is Hausdorff.